Homework Set 1

Due: September 23, 2008, *before class*

1. **Energy Generation**

   (a) **Compute the specific energy generation rate of the Sun as a whole.**

   $$\varepsilon_{\text{sun}} = \frac{1\,\mathrm{L}_\odot}{1\,\mathrm{M}_\odot} = \frac{3.83 \times 10^{33}\,\mathrm{erg\,s^{-1}}}{1.9891 \times 10^{33}\,\mathrm{g}} = 1.93\,\mathrm{erg\,g^{-1}\,s^{-1}}$$

   Score: **2**

   (b) **Assume a human of weight 100 kg has a "luminosity" of 100 W. Compute the specific energy generation rate of a this human.**

   $$\varepsilon_{\text{human}} = \frac{10^9\,\mathrm{erg\,s^{-1}}}{10^5\,\mathrm{g}} = 10^4\,\mathrm{erg\,g^{-1}\,s^{-1}}$$

   Score: **2**

   (c) **Compare the results.**
   The specific energy generation of a human is about 5,000 times that of the sun. Score: **1**

   (d) Modern micoprocessors have now reached a "gate width" of 45 nm. Assume this corresponds to the thickness of the "active" layer that contains micoprocessors and density of silicon of 2.33 g/cm^3. The typical die size is about 100 mm^2 and they have a power up about 100 W.
   What is the specific energy generation rate of the active layer?

   $$\varepsilon_{\text{CPU}} = \frac{10^9\,\mathrm{erg\,s^{-1}}}{4.5 \times 10^{-6}\,\mathrm{cm} \times 1\,\mathrm{cm^2} \times 2.33\,\mathrm{g\,cm^{-3}}} = 9.53 \times 10^{13}\,\mathrm{erg\,g^{-1}\,s^{-1}}$$

   Score: **4**

   (e) **How long does it take for the "active" layer of the CPU to release as much energy as their rest mass?**

   $$\tau = \frac{c^2}{\varepsilon_{\text{CPU}}} = 9.4 \times 10^6\,\mathrm{s} = 2,600\,\mathrm{h} = 109\,\mathrm{d} = 0.30\,\mathrm{yr}$$

   Score: **4**

   (f) **What happens when you run your computer that long – having converted the rest mass of the "active" layer into energy?**
   Well, the CPU does not dissipate internal energy or its own rest mass, but rather electricity supplied from the outside. But, yes, that electricity comes from rest mass somewhere else, e.g., stored in nuclear or chemical form, or in gravitational potential energy. Score: **4**

   (g) **What is the specific energy generation rate corresponding to an element of mass radiating away its entire rest mass in 1 s?**

   $$\varepsilon = \frac{c^2}{1\,\mathrm{s}} = 8.91 \times 10^{20}\,\mathrm{erg\,g^{-1}\,s^{-1}}$$

   Score: **2**

(h) Assume a characteristic chemical energy content of 10 eV per nucleon, and a characteristic nuclear energy content of 10 MeV per nucleon.
Compute the energy content (supply) of the sun for each of these assumptions. How long could the sun shine at its current luminosity from each of these energy sources?

$$E_{\mathrm{chemical},\odot} = 10\,\mathrm{eV} \times N_\mathrm{A} \times \mathrm{M}_\odot = 1.92 \times 10^{46}\,\mathrm{erg}$$
$$E_{\mathrm{nuclear},\odot} = 10\,\mathrm{MeV} \times N_\mathrm{A} \times \mathrm{M}_\odot = 1.92 \times 10^{52}\,\mathrm{erg}$$
$$\tau_{\mathrm{chemical},\odot} = E_{\mathrm{chemical},\odot}/\mathrm{L}_\odot = 5.01 \times 10^{12}\ \mathrm{sec} = 1.59 \times 10^{5}\,\mathrm{yr}$$
$$\tau_{\mathrm{nuclear},\odot} = E_{\mathrm{nuclear},\odot}/\mathrm{L}_\odot = 5.01 \times 10^{18}\ \mathrm{sec} = 1.59 \times 10^{11}\,\mathrm{yr}$$

Score: **2**

2. **Nuclear Reaction Kinematics**
Based on the general formula for nuclear reactions,

$$\frac{\partial}{\partial t} Y_i = \sum_{\substack{\alpha_1,\alpha_2,\ldots \\ \beta_1,\beta_2,\ldots}} \lambda_{\alpha_1\mathbf{1}+\alpha_2\mathbf{2}+\ldots\rightarrow\beta_1\mathbf{1}+\beta_2\mathbf{2}+\ldots} \frac{\beta_i - \alpha_i}{\alpha_1!\alpha_2!\ldots} Y_1^{\alpha_1} Y_2^{\alpha_2}\ldots$$

(a) **Write the equation for the change of hydrogen, $\frac{\partial}{\partial t}Y_{^1\mathrm{H}}$, in the reaction of the last step of the CNO-1 cycle for hydrogen bunring, ^{15}N + ^{1}H $\longmapsto$ ^{12}C +^{4}He (simple binary reaction prototype).**

$$\frac{\partial}{\partial t} Y_{^1\mathrm{H}} = -\lambda_{^1\mathrm{H}+^{15}\mathrm{N}\rightarrow^{12}\mathrm{C}+^4\mathrm{He}} Y_{^1\mathrm{H}} Y_{^{15}\mathrm{N}}$$

Score: **2**

(b) **The system of equations for the changes $\frac{\partial}{\partial t}Y_i$ of ($i =$) ^{1}H, ^{2}H, ^{3}He, and ^{4}He due to the pp1 chain for hydrogen burning. Assume a net production of 1 (one) ^{4}He nucleus from 4 (four) ^{1}H nuclei. Assume the reaction is in equilibrium (steady state).**

$$\frac{\partial}{\partial t} Y_{^1\mathrm{H}} = -\lambda_{2\,^1\mathrm{H}\rightarrow^2\mathrm{H}}\,(Y_{^1\mathrm{H}})^2 - \lambda_{^1\mathrm{H}+^2\mathrm{H}\rightarrow^3\mathrm{He}}\,Y_{^1\mathrm{H}}Y_{^2\mathrm{H}} + \lambda_{2\,^3\mathrm{He}\rightarrow^4\mathrm{He}+2\,^1\mathrm{H}}\,(Y_{^3\mathrm{He}})^2 \quad (1)$$
$$\frac{\partial}{\partial t} Y_{^2\mathrm{H}} = +\lambda_{2\,^1\mathrm{H}\rightarrow^2\mathrm{H}}\,\frac{1}{2}(Y_{^1\mathrm{H}})^2 - \lambda_{^1\mathrm{H}+^2\mathrm{H}\rightarrow^3\mathrm{He}}\,Y_{^1\mathrm{H}}Y_{^2\mathrm{H}} \quad (2)$$
$$\frac{\partial}{\partial t} Y_{^3\mathrm{He}} = +\lambda_{^1\mathrm{H}+^2\mathrm{H}\rightarrow^3\mathrm{He}}\,Y_{^1\mathrm{H}}Y_{^2\mathrm{H}} - \lambda_{2\,^3\mathrm{He}\rightarrow^4\mathrm{He}+2\,^1\mathrm{H}}\,(Y_{^3\mathrm{He}})^2 \quad (3)$$
$$\frac{\partial}{\partial t} Y_{^4\mathrm{He}} = +\lambda_{2\,^3\mathrm{He}\rightarrow^4\mathrm{He}+2\,^1\mathrm{H}}\,\frac{1}{2}(Y_{^3\mathrm{He}})^2 \quad (4)$$

Score: **8**

(c) **Assuming steady state, what are the timely changes of ^{2}H and ^{3}He?**

$$\frac{\partial}{\partial t} Y_{^2\mathrm{H}} = 0 = \frac{\partial}{\partial t} Y_{^3\mathrm{He}}$$

Score: **2**

(d) **Express the abundance of ^{2}H in terms of that of ^{1}H and the relevant reaction rates λ_i.**
From Eq. 2, and $\frac{\partial}{\partial t}Y_{^2\mathrm{H}}$ we can solve for $Y_{^2\mathrm{H}}$:

$$Y_{^2\mathrm{H}} = \frac{Y_{^1\mathrm{H}}}{2}\,\frac{\lambda_{2\,^1\mathrm{H}\rightarrow^2\mathrm{H}}}{\lambda_{^1\mathrm{H}+^2\mathrm{H}\rightarrow^3\mathrm{He}}}$$

Score: **4**

(e) **Add specific values for the lambda's.**
Assume a central temperature of sun about 1.6×10^7 K, a central density about $160\,\mathrm{g\,cm^{-3}}$, and that the sun has burnt half of its ^{1}H fuel by now.
Note: For a binary reaction we have $\lambda = N_\mathrm{A} \langle \sigma v \rangle \rho$
Obtain values for nuclear reaction rates from
`http://www.phy.ornl.gov/astrophysics/data/cf88/nuclei.html`

Using
$\lambda_{2\,^1\mathrm{H} \to ^2\mathrm{H}} = 1.05 \times 10^{-19} \times 160\,\mathrm{g\,mol^{-1}\,s^{-1}}$,
$\lambda_{^1\mathrm{H}+^2\mathrm{H} \to ^3\mathrm{He}} = 1.76 \times 10^{-2} \times 160\,\mathrm{g\,mol^{-1}\,s^{-1}}$,
$Y_{^1\mathrm{H}} = 0.35$, we obtain

$$Y_{^2\mathrm{H}} = \frac{0.35 \times 1.05 \times 10^{-19}}{2 \times 1.76 \times 10^{-2}} = 1.04 \times 10^{-18}$$

Score: **4**

3. **Nuclear Reaction Rates**
Based on the general dependence of a non-resonant binary nuclear reaction,

$$\langle \sigma v \rangle \propto (kT)^{-2/3}\, e^{-\frac{3}{2}\left(\frac{4\pi^2 Z_1 Z_2 e^2}{h}\right)^{2/3}\left(\frac{m_\mathrm{red}}{kT}\right)^{1/3}}$$

compute the temperature sensitivity of carbon burning, ^{12}C+^{12}C at $T = 10^9$ K, that is, compute the exponent n in

$$\langle \sigma v \rangle \propto T^n$$

where n is given by

$$n = \frac{\mathrm{d} \ln \langle \sigma v \rangle}{\mathrm{d} \ln T}\ .$$

Solution:

$$\ln \langle \sigma v \rangle = -\frac{2}{3} \ln T - C T^{-1/3}\ ,$$

$$C := \frac{3}{2}\left[\left(\frac{4\pi^2 Z_1 Z_2 e^2}{h}\right)^2 \frac{m_\mathrm{red}}{k_\mathrm{B}}\right]^{1/3}$$

$$m_\mathrm{red} = \left(\frac{1\,\mathrm{g}}{N_\mathrm{A}}\right) \frac{A_1 A_2}{A_1 + A_2}$$

$$T \frac{\mathrm{d}}{\mathrm{d}T} \ln \langle \sigma v \rangle = -\frac{2}{3} + \frac{C}{3} T^{-1/3}$$

Defining $\tau := C T^{-1/3}$ and $T_9 = T/10^9$ K we obtain

$$\tau = 4.2487 \times \left(Z_1^2 Z_2^2 \frac{A_1 A_2}{A_1 + A_2} \frac{1}{T_9}\right)^{1/3}$$

$$n = \frac{\tau - 2}{3}$$

For the specific case $A_1 = A_2 = 12$, $Z_1 = Z_2 = 6$, $T_9 = 1$,

$$\tau = 4.2487 \times \left(6^2 \times 6^2 \frac{12 \times 12}{12 + 12}\right)^{1/3} = 4.2487 \times (7776)^{1/3} = 84.173$$

$$n = \frac{84.173 - 2}{3} = 27.391$$

Score: **8**

4. **Stellar Evolution Project**
   download Bill Paxton's **EZ Stellar Evolution** code
   `http://www.kitp.ucsb.edu/~paxton/EZ-intro.html`

   The code uses `gfortran` (Linux, MacOS).

   There is also the **g95** FORTRAN compiler can be downloaded for most platforms.
   `http://www.g95.org`