Homework Set 2

Due: October 7, 2008, *before class*

1. **Energy Generation**

A table of mass excess is given at
`http://webusers.physics.umn.edu/~alex/stellarevolution/AST-4001/homework/mass_table.txt`
The table lists in each line Z, A, and mass excess per nucleus in MeV.

(a) Compute the mass excess and binding energy (per nucleon).
Which is the most tightly bound nucleus (binding energy per nucleon) and which has the highest mass excess *per nucleon*?
The nucleus with the highest total mass excess in the Table is $^{338}\mathsf{No}$.
The nucleus with the highest mass excess per nucleon is the neutron, but that with the lowest mass excess *per nucleon* is $^{56}\mathsf{Fe}$. It is the most stable nucleus.

$$\mathrm{BE} = \mathrm{ME}(p) \times n_p + \mathrm{ME}(n) \times n_n - \mathrm{ME}(\mathrm{nucleus})$$

The nucleus with the highest total binding energy listed is $^{339}126$ (unbihexium), but the one with the highest binding energy *per nucleon* is $^{52}\mathsf{Cr}$, followed by $^{56}\mathsf{Fe}$ and $^{62}\mathsf{Ni}$. Score: **4+2**

(b) **For the conditions in the center of the sun, central T about $1.6\times10^{7}\,\mathsf{K}$, central density about $160\,\mathsf{g\,cm^{-3}}$, find for the reaction $^{3}\mathsf{He} + {}^{3}\mathsf{He} \rightarrow {}^{4}\mathsf{He} + {}^{1}\mathsf{H} + {}^{1}\mathsf{H}$ the reaction rate at**
`http://www.phy.ornl.gov/astrophysics/data/cf88/nuclei.html;`
and then determine the nuclear energy generation rate from the mass excess of the nuclei.
The tabulated value is 6.45×10^{-10}. mass excesses are:
$^{1}\mathsf{H}$: $7.28897\,\mathsf{MeV}$,
$^{3}\mathsf{He}$: $14.9312\,\mathsf{MeV}$
$^{4}\mathsf{He}$: $2.42492\,\mathsf{MeV}$
Per reaction we release
$(2 \times 14.9312 - 2 \times 7.28897 - 2.42492)\,\mathsf{MeV} = 12.85954\,\mathsf{MeV}$
The reaction rate is proportional to ρ^2 and $Y^2_{^3\mathsf{He}}$ and hence given by
$r = \frac{1}{2}6.45\times10^{-10} \times 160 \times Y^2_{^3\mathsf{He}}$.
The nuclear energy generation rate is then

$$\epsilon = 12.85942\,\mathsf{MeV} \times 5.16\times10^{-8}/\mathsf{s}\ N_\mathsf{A}\ Y^2_{^3\mathsf{He}} = 6.4\times10^{11}\,\mathsf{erg/s/g}\ Y^2_{^3\mathsf{He}}$$

Score: **6**

2. **Stars and Heat**

   (a) Assume an ideal gas and an average temperature of for a hydrostatic stars (Prialnik, Chapter 2.4).
   Determine the heat capacity, C, of star,

   $$C = \frac{\mathsf{d}E}{\mathsf{d}\bar{T}}$$

   From the Virial Theorem we use that $E = -U$. From Prialnik equation 2.28 we use

   $$U = \frac{3k_\mathsf{B}}{2m_g}\bar{T}M$$

   Hence we have

   $$C = -\frac{3k_\mathsf{B}}{2m_g}M$$

   That is, a *negative* heat capacity! Score: **4**

   (b) **How does this change for an ideal gas with radiation, assuming constant β?**
   Following Prialnik exercise 3.2 we have

   $$E = -\frac{\beta}{2-\beta}U\,.$$

   The internal heat is

   $$U = \frac{3}{2}(2-\beta)\frac{1}{m_g}\bar{T}M$$

   and hence

   $$C = -\beta\frac{3k_\mathsf{B}}{2m_g}M$$

   That is, for gas with radiation the absolute value of the heat capacity decreases. Score: **4**

   (c) **What happens if you "cover" the star (no radiation is allowed to escape from the surface) and assume nuclear burning would continue at a constant specific rate.**
   The *total* energy of the star would *increase* – its absolute value would decrease – and therefore, because of the negative heat capacity, the temperature would decrease. Score: **4**

3. **Stellar Stability**

For and ideal gas with radiation, the condition for *local* stability is given by

$$\left(\frac{\mathsf{d}\ln T}{\mathsf{d}\ln P}\right)_s < \left(\frac{\mathsf{d}\ln T}{\mathsf{d}\ln P}\right)_e + \frac{\varphi}{\delta}\left(\frac{\mathsf{d}\ln \mu}{\mathsf{d}\ln P}\right)_s$$

where the index "s" refers to the stratification in the surroundings, "e" to that of an adiabatically displaced element that does not mix with it surroundings and remains fully ionized.

$$\delta := -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_{P,\mu}, \qquad \varphi = \left(\frac{\partial \ln \rho}{\partial \ln \mu}\right)_{P,T}$$

(a) **How does this relation simplify if there is no composition gradient?**

$$\left(\frac{\mathsf{d}\ln T}{\mathsf{d}\ln P}\right)_s < \left(\frac{\mathsf{d}\ln T}{\mathsf{d}\ln P}\right)_e$$

Score: **2**

(b) **Derive a similar relation for a prefect non-relativistic completely degenerate gas.**
Hint: in this case the EOS does not depend on μ, but on ...

... mean molecular weight per electron, μ_e

We obtain

$$\delta\left(\frac{\mathsf{d}\ln T}{\mathsf{d}\ln P}\right)_s < \delta\left(\frac{\mathsf{d}\ln T}{\mathsf{d}\ln P}\right)_e + \varphi\left(\frac{\mathsf{d}\ln \mu_e}{\mathsf{d}\ln P}\right)_s$$

with

$$\varphi = \left(\frac{\partial \ln \rho}{\partial \ln \mu_e}\right)_{P,T}$$

and

$$\delta := -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_{P,\mu_e}$$

In the next subquestion we will see that actually $\delta = 0$ and hence

$$0 < \varphi\left(\frac{\mathsf{d}\ln \mu_e}{\mathsf{d}\ln P}\right)_s$$

Score: **4+2**

(c) **Compute δ and φ for the above case.**

From equation 3.33 of Prialnik we have

$$P_{\mathrm{e,deg}} = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3} m_{\mathrm{H}}^{-5/3}\left(\frac{\rho}{\mu_e}\right)^{5/3}$$

Since for constant P here ρ is independent of T we have

$$\delta := -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_{P,\mu_e} = 0$$

Also, for a given P here ρ is proportional to μ_e and therefore

$$\varphi = \left(\frac{\partial \ln \rho}{\partial \ln \mu_e}\right)_{P,T} = 1$$

Score: **4**

4. **The Eddington Limit**

The Eddington limit is reached when, at the surface of the star, the acceleration due to the radiation pressure gradient balances gravitational acceleration.

(a) **Derive the Eddington Luminosity based on this statement.**

The radiation pressure is given by $P_{\mathsf{rad}} = \frac{a}{3}T^4$, hence its gradient is

$$\frac{\mathsf{d}P}{\mathsf{d}r} = \frac{4a}{3}T^3\frac{\mathsf{d}T}{\mathsf{d}r}$$

The temperature gradient is given by (Prialnik 3.69)

$$\frac{\mathsf{d}T}{\mathsf{d}r} = -\frac{3\kappa L}{4acT^3 4\pi r^2}$$

and we get

$$\frac{\mathsf{d}P}{\mathsf{d}r} = -\frac{\kappa\rho L}{4\pi r^2 c}$$

Comparing this to the gravitational acceleration, (force per unit volume)

$$\frac{\mathsf{d}P}{\mathsf{d}r} = -\rho\frac{GM}{r^2}$$

and solving for L we obtain

$$L_{\mathsf{edd}} = \frac{4\pi cGM}{\kappa}$$

Score: **4**

(b) For a mixture of hydrogen and helium, the electron scattering opacity is given by

$$\kappa_{\mathrm{es}} = (1+X)\times 0.2\,\mathsf{cm}^2/g$$

where X is the hydrogen mass fraction. For the sun X is about 70 % at the surface.

Assuming the opacity at the surface of the sun where just due to electron scattering, compute the Eddington Luminosity of the sun.

$$L_{\mathsf{edd},\odot} = \frac{4\pi cG\mathsf{M}_\odot}{(1+0.7)\times 0.2\,\mathsf{cm}^2/\mathsf{g}} = \frac{4\pi cG\mathsf{M}_\odot}{0.34\,\mathsf{cm}^2/\mathsf{g}} = 5\times 10^{37}/0.34\,\mathsf{erg\,s}^{-1}$$

$$\ldots = 1.47\times 10^{38}\,\mathsf{erg\,s}^{-1} = 1.47\times 10^{31}\,\mathrm{W} = 3.84\times 10^4\,\mathsf{L}_\odot$$

Score: **2**

(c) **Based on last week's homework assignment, what is the Eddington luminosity of a human and how does it compare to the assumed "luminosity"?**

$M = 100\,\mathsf{kg} = 1\times 10^5\,\mathsf{g}$, hydrogen mass fraction is 10 % and we assume just electron scattering, though surely not realistic

$$L_{\mathsf{edd},\odot} = \frac{4\pi cG\,10^5\,\mathsf{g}}{(1+0.1)\times 0.2\,\mathsf{cm}^2/\mathsf{g}} = \frac{4\pi cG\,10^5\,\mathsf{g}}{0.22\,\mathsf{cm}^2/\mathsf{g}} = 1.1\times 10^{10}\mathsf{erg\,s}^{-1}$$

This is to be compared to 100 W$= 10^9\,\mathsf{erg\,s}^{-1}$. Humans "shine" at about 10 % Eddington luminosity (only considering electron scattering opacity).

Score: **2**

(d) **If, very naively, one assumed a star did not deform due to rapid rotation, how is the Eddington limit (luminosity) modified, at the equator, for a rotating star?**

One has to take into account the centrifugal force, $\omega^2 r$ which effectively reduces gravity. Here ω is the angular velocity of the stellar rotation. Hence the "effective" Eddington luminosity is reduced.
If we replace in the above derivation

$$-\rho \frac{GM}{r^2}$$

by

$$\rho \left(-\frac{GM}{r^2} + \omega^2 r \right)$$

and define the ratio of centrifugal force to gravitational acceleration at the equator as

$$\Omega = \omega^2 r \left(\frac{GM}{r^2} \right)^{-1} = \frac{\omega^2 r^3}{GM}$$

we obtain a "modified Eddington Limit"

$$L_{\mathsf{edd}} = \frac{4\pi c GM}{\kappa} (1 - \Omega) \ .$$

Score: **6**

(e) **What happens at the pole?**

In this simple approximation nothing happens at the pole.
(Eventually deformation of the star will have an effect on the pole as well.) Score: **2**

(f) **If a star cannot exceed the Eddington Luminosity corrected for centrifugal force at the equator (above question and assumptions), derive a maximum rotation rate for the star.**

The maximum velocity is reached when $\Omega = 1$. Therefore the maximum rotation rate is

$$\omega = \sqrt{GM/r^3}$$

Score: **4**

5. **Convection**

Assume a chemically homogeneous convection zone of thickness r_c and ideal gas. A bubble with temperature excess ΔT. Assume the bubble has an initial velocity of zero, then starts rising and maintains a constant $\Delta T/T$ (adiabatic stratification and adiabatic expansion of the bubble). The bottom of the convection zone is at R_0. Assume there is no drag. Neglect the mass in the convection zone.

(a) Assume $R_0 \gg r_c$ (plane parallel approximation).
Compute the time average velocity of the bubble.
Gravitational acceleration g is constant in this approximation.
The density perturbation is given by

$$\frac{\Delta\rho}{\rho} = -\delta\frac{\Delta T}{T}$$

The acceleration a due to buoyancy is given by

$$a = -g\frac{\Delta\rho}{\rho} = g\delta\frac{\Delta T}{T}$$

We also use the equations of motion for constant acceleration

$$v(t) = at\ , \quad r(t) = \frac{1}{2}at^2$$

The time τ for a bubble to reach the top is then given by

$$\tau = \sqrt{2r_c/a}$$

And the time average velocity by

$$\langle v\rangle_t = \int_0^\tau v(t)\mathrm{d}t\Big/\int_0^\tau \mathrm{d}t = \frac{1}{2}a\tau^2/\tau = \frac{1}{2}a\tau = \sqrt{ar_c/2} = \sqrt{r_c g\delta\frac{\Delta T}{2T}}$$

Alternatively one could also argue that for constant acceleration the average velocity is just half of the velocity at the top ($a\tau$) and hence obtain $\langle v\rangle_t = \frac{1}{2}a\tau$ this way. Or, even easier, that it is distance divided by the time required to reach the top, $\langle v\rangle_t = r_c/\tau = \sqrt{ar_c/2}$.
For an ideal gas $\delta = 1$.

Score: **8**

(b) **Compute the spatial average velocity of the bubble.**
From the above equations the time to reach a distance r is given by

$$t(r) = \sqrt{2r_c/a}$$

Using this we can now compute the velocity as a function or time

$$v(r) = v(t(r)) = a\sqrt{2r_c/a} = \sqrt{2ar_c}$$

The spatial average velocity is then given by

$$\langle v\rangle_r = \int_0^{r_c}\sqrt{2ar_c}\mathrm{d}r\Big/\int_0^{r_c}\mathrm{d}r = \sqrt{2a}\frac{2}{3}r_c^{3/2}/r_c = \frac{2}{3}\sqrt{2ar_c} = \frac{4}{3}\langle v\rangle_t = \frac{4}{3}\sqrt{r_c g\delta\frac{\Delta T}{2T}} = \sqrt{r_c g\delta\frac{8\Delta T}{9T}}$$

Score: **4**

(c) Now assume that $R_0 = 1\,\mathsf{R}_\odot$, $r_c = 1000\,\mathsf{R}_\odot$ (point mass - red supergiant).
Compute the time average velocity of the bubble.
This can be solved analytically. If you evaluate the integral numerically, please document how it was done.

This is best solved from the energy equation taking into account buoyancy.
At the bottom we have a specific energy and acceleration of the bubble of

$$E_0 = -\frac{GM}{R_0}\delta\frac{\Delta T}{T}\ , \quad a_0 = \frac{GM}{R_0^2}\delta\frac{\Delta T}{T}$$

where M is the mass of the star. As before, the mass in the envelope is neglected. Further out in the star we have

$$E(r) = -\frac{GM}{r}\delta\frac{\Delta T}{T} = E_0\frac{R_0}{r}$$

The energy difference $E(r) - E_0$ has to be equal to the specific kinetic energy $v^2/2$ and we can solve for $v(r)$:

$$v(r) = \sqrt{2\left(E(r) - E_0\right)} = \frac{\mathsf{d}r}{\mathsf{d}t}$$

We now do a variable separation and solve for the time to reach the top

$$\tau = \int_0^\tau \mathsf{d}t = \int_{R_0}^{R_0+r_c} \frac{1}{\sqrt{2\left(E(r) - E_0\right)}}\mathsf{d}r = \frac{1}{\sqrt{-2E_0}}\int_{R_0}^{R_0+r_c} \frac{1}{\sqrt{\left(1-\frac{R_0}{r}\right)}}\mathsf{d}r$$

(Note that E_0 is less than 0.) Introducing $\xi = r/R_0$ we can write

$$\tau = \frac{R_0}{\sqrt{-2E_0}}\int_{\xi_0}^{\xi_1} \frac{1}{\sqrt{(1-1/\xi)}}\mathsf{d}\xi = \frac{R_0}{\sqrt{-2E_0}}\int_{\xi_0}^{\xi_1} \sqrt{\frac{\xi}{\xi-1}}\mathsf{d}\xi$$

with $\xi_0 = 1$, $\xi_1 = (R_0 + r_c)/R_0 = 1001$. Using

$$\int\sqrt{\frac{x}{a+x}}dx = \sqrt{x(a+x)} - a\ln\left(\sqrt{x} + \sqrt{x+a}\right) + C$$

with $a = -1$, $x = \xi$ and our integration boundaries we obtain for the integral

$$\int_{\xi_0}^{\xi_1}\sqrt{\frac{\xi}{\xi-1}}\mathsf{d}\xi = \left[\sqrt{\xi(\xi-1)} + \ln\left(\sqrt{\xi} + \sqrt{\xi-1}\right)\right]_{\xi_0}^{\xi_1}$$

$$\ldots = \left[\sqrt{\xi(\xi-1)} + \ln\left(\sqrt{\xi} + \sqrt{\xi-1}\right)\right]_{1}^{1001}$$

$$\ldots = \sqrt{1001000} - \sqrt{0} + \ln\left(\sqrt{1001} + \sqrt{1000}\right) - \ln(1)$$

$$\ldots = \sqrt{1001000} + \ln\left(\sqrt{1001} + \sqrt{1000}\right)$$

$$\ldots = 1000.5 + 4.147 = 1003.647$$

And hence

$$\tau = 1004.647\frac{R_0}{\sqrt{-2E_0}} = 710.393\frac{R_0}{\sqrt{-E_0}}$$

The average velocity is then

$$\langle v\rangle_t = \frac{r_c}{\tau} = \frac{1000}{710.393}\sqrt{-E_0} = 1.408\sqrt{-E_0} = 1.408\sqrt{\frac{GM}{R_0}\delta\frac{\Delta T}{T}} \approx \sqrt{2a_0R_0}$$

Score: **12**