Homework Set 4

Due: November 4, 2008, *before class*

1. **Initial Mass Function**

The number of stars in a mass bin $[M, M+\mathrm{d}M]$ can be written defining the birth function $\Phi(M)$:

$$\mathrm{d}N = \Phi(M)\mathrm{d}M$$

The mass of stars in a mass bin is then given by weighing by mass M, defining the initial mass function (**IMF**) $\xi(M)$:

$$\xi(M) = M\mathrm{d}N/\mathrm{d}M$$

Salpeter (1955) found observationally a power law for Φ, ξ:

$$\Phi(M) \propto M^{-2.35}\ , \quad \xi(M) \propto M^{-1.35}$$

As a lower mass limit for stars (start of hydrostatic hydrogen burning) usually a value of $0.08\,\mathrm{M}_\odot$ is assumed. Further, assume that stars above $9\,\mathrm{M}_\odot$ make supernovae, below that they make white dwarfs. Stars below $7.5\,\mathrm{M}_\odot$ make CO white dwarfs. Further, assume that massive stars above $25\,\mathrm{M}_\odot$ make black holes, below that they make neutron stars. A conservative upper mass limit for present-day stars may be around $150\,\mathrm{M}_\odot$. Stars below $2\,\mathrm{M}_\odot$ are considered low-mass stars.

(a) **Based on these relations and parameters, what fraction of stars make supernovae?**

Total number of stars between M_0 and M_1:

$$N = C\int_{M_0}^{M_1}\Phi(M)\mathrm{d}M = C\frac{1}{-1.35}\left[M^{-1.35}\right]_{M_0}^{M_1} = C\frac{1}{-1.35}\left(M_1^{-1.35} - M_0^{-1.35}\right)$$

$$\ldots = C\frac{1}{1.35}\left(M_0^{-1.35} - M_1^{-1.35}\right) \propto \left(M_0^{-1.35} - M_1^{-1.35}\right)$$

where C is an arbitrary constant to scale to the actual number of stars. The total number of stars (defining $C' = C/(1.35\,\mathrm{M}_\odot{}^{1.35})$):

$$N = C'\left((0.08)^{-1.35} - (150)^{-1.35}\right) = C'(30.25718478 - 0.00115419) = C' \times 30.25603059$$

number of stars that make supernovae

$$N_{\mathrm{SN}} = C'\left((9)^{-1.35} - (150)^{-1.35}\right) = C' \times (0.05149590 - 0.00115419) = C' \times 0.05034171$$

Fraction of stars that make supernovae

$$N_{\mathrm{SN}}/N = 0.05034171/30.25603059 = 1.66\times10^{-3} = 0.166\,\%$$

Score: **4**

(b) **What fraction of supernovae come from stars in the mass range 9-10 solar masses?**

Number of stars in the $9\ldots10\,\mathrm{M}_\odot$ mass range:

$$N_{9-10} = C'\left((9)^{-1.35} - (10)^{-1.35}\right) = C' \times (0.05149590 - 0.04466836) = C' \times 0.00682754$$

$$N_{9-10}/N_{\mathrm{SN}} = 0.00682754/0.05034171 = 0.1356 = 13.56\,\%$$

Score: **2**

(c) **What fractions of stars make CO white dwarfs, make NeMgO white dwarfs, neutron stars, and black holes?**

$$N_{\rm CO} = C'\left((0.08)^{-1.35} - (7.5)^{-1.35}\right) = C' \times (30.2572 - 0.06586691) = C' \times 30.1913$$

$$N_{\rm CO}/N = 0.997861 \approx 99.79\,\%$$

$$N_{\rm NeMgO} = C'\left((7.5)^{-1.35} - (9)^{-1.35}\right) = C' \times (0.06586691 - 0.05149590) = C' \times 0.01437101$$

$$N_{\rm NeMgO}/N = 4.750{\times}10^{-4} \approx 0.05\,\%$$

$$N_{\rm NS} = C'\left((9)^{-1.35} - (25)^{-1.35}\right) = C' \times (0.05149590 - 0.01296525) = C' \times 0.03853064$$

$$N_{\rm NS}/N = 1.273{\times}10^{-3} \approx 0.1\,\%$$

$$N_{\rm BH} = C'\left((25)^{-1.35} - (150)^{-1.35}\right) = C' \times (0.01296525 - 0.00115419) = C' \times 0.01811064$$

$$N_{\rm BH}/N = 3.903{\times}10^{-4} \approx 0.04\,\%$$

Score: **8**

(d) **Compare the *mass* that goes in massive stars, intermediate mass stars, and low-mass stars.**

Total mass of stars between M_0 and M_1:

$$N = D\int_{M_0}^{M_1} \xi(M)\mathrm{d}M = D\frac{1}{-0.35}\left[M^{-0.35}\right]_{M_0}^{M_1} = D\frac{1}{-0.35}\left(M_1^{-0.35} - M_0^{-0.35}\right)$$

$$\ldots = D\frac{1}{0.35}\left(M_0^{-0.35} - M_1^{-0.35}\right) \propto \left(M_0^{-0.35} - M_1^{-0.35}\right)$$

where D is an arbitrary constant to scale to the actual total mass of stars. Defining $D' = D/(0.35\,\mathrm{M}_\odot{}^{0.35})$ one obtains

$$M_{\rm massive} = D'\left((9)^{-0.35} - (150)^{-0.35}\right) = D' \times (0.45346306 - 0.17312830) = D' \times 0.2903$$

$$M_{\rm intermediate} = D'\left((2)^{-0.35} - (9)^{-0.35}\right) = D' \times (0.78458410 - 0.45346306) = D' \times 0.3311$$

$$M_{\rm low-mass} = D'\left((0.08)^{-0.35} - (2)^{-0.35}\right) = D' \times (2.42057478 - 0.78458410) = D' \times 1.6360$$

Hence the ratio of mass that goes into these different stellar mass regime is given by

$$M_{\rm massive} : M_{\rm intermediate} : M_{\rm low-mass} = 0.2903 : 0.3311 : 1.6360 \approx 0.1292 : 0.1429 : 0.7279$$

Score: **8**

2. **Cluster Ages**

If the lifetime of a star is approximated from assuming it burns up half of its fuel, an 25 solar mass stars lives about 10^7 yr, and $L \propto M^2$, what are the most massive stars that are still around after 10^8 yr?

$$L \propto M^2\,, \quad E \propto M\,, \quad \tau \propto E/L \propto 1/M$$

$$\frac{M}{25\,\mathrm{M}_\odot} = \frac{\tau_{25\,\mathrm{M}_\odot}}{10^8\,\mathrm{yr}} = \frac{10^7\,\mathrm{yr}}{10^8\,\mathrm{yr}} = \frac{1}{10}$$

$$\Rightarrow M = 2.5\,\mathrm{M}_\odot$$

Score: **4**

3. **Lane-Emden Equation**

Derive a formula for B_n (formula 5.28 of Prialnik) that contains all the n-dependent terms as a function of M_n, R_n and D_n. Check you result with Table 5.1

From the Lane-Emden equation we have

$$P_{\mathsf{c}} = \frac{(4\pi G)^{\frac{1}{n}}}{1+n}\left(\frac{GM}{M_n}\right)^{\frac{n-1}{n}}\left(\frac{R}{R_n}\right)^{\frac{3-n}{n}}\rho_{\mathsf{c}}^{\frac{n+1}{n}}.$$

Taking this to power n for simplicity, we have

$$P_{\mathsf{c}}^n = \frac{(4\pi G)}{(1+n)^n}\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n}\rho_{\mathsf{c}}^{n+1}.$$

We now eliminate R from the Lame-Emden relation for ρ_{c},

$$\rho_{\mathsf{c}} = D_n\frac{3M}{4\pi R^3} \quad\Rightarrow\quad R = \left(\frac{3MD_n}{4\pi\rho_{\mathsf{c}}}\right)^{\frac{1}{3}}$$

and obtain

$$P_{\mathsf{c}}^n = \frac{(4\pi G)}{(1+n)^n}\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{1}{R_n}\right)^{3-n}\left(\frac{3MD_n}{4\pi\rho_{\mathsf{c}}}\right)^{\frac{3-n}{3}}\rho_{\mathsf{c}}^{n+1}.$$

Collecting terms we obtain

$$P_{\mathsf{c}}^n = \frac{(4\pi)^{\frac{n}{3}}G^n}{(1+n)^n}M_n^{1-n}M^{\frac{2n}{3}}R_n{}^{n-3}(3D_n)^{\frac{3-n}{3}}\rho_{\mathsf{c}}^{\frac{4n}{3}}$$

and we can write

$$P_{\mathsf{c}} = (4\pi)^{\frac{1}{3}}GM^{\frac{2}{3}}\rho_{\mathsf{c}}^{\frac{4}{3}}\left[M_n^{\frac{1-n}{n}}R_n^{\frac{n-3}{n}}(3D_n)^{\frac{3-n}{3n}}\frac{1}{1+n}\right]$$

$$P_{\mathsf{c}} = (4\pi)^{\frac{1}{3}}B_nGM^{\frac{2}{3}}\rho_{\mathsf{c}}^{\frac{4}{3}}$$

with

$$B_n = M_n^{\frac{1-n}{n}}R_n^{\frac{n-3}{n}}(3D_n)^{\frac{3-n}{3n}}\frac{1}{1+n}.$$

Score: **8**

4. **Main Sequence**

Using the homolgy relation derived for Main-Sequence stars, compare the luminosity of a star made of pure helium and one made of pure hydrogen. Neglect the fact that nuclear burning would proceed differently in these stars.

Using

$$F_* = \frac{ac}{\kappa}\left(\frac{\mu G}{\mathcal{R}}\right)^4 M^3$$

we know that $L \propto F_*$.
We have in principle two things that change:

- mean molecular weight μ
- opacity κ

The mean molecular weight for fully ionized pure hydrogen is $\mu_{\mathsf{H}} = 0.5$, for fully ionized pure helium it is $\mu_{\mathsf{He}} = 4/3$. With

$$F_* \propto \mu^4$$

we hence fine that the luminosity of a helium star would be brighter by a factor

$$\left(\frac{\mu_{\mathsf{He}}}{\mu_{\mathsf{H}}}\right)^4 = \left(\frac{8}{3}\right)^4 = 50.5679$$

We may also consider the change of opacity, here for simplicity assuming pure Thompson electron scattering,

$$\kappa = \kappa_{\mathsf{es}}(X) = \kappa_{\mathsf{es},0}\,\frac{1}{2}(1+X) = \frac{\sigma_{\mathsf{T}}}{\mathsf{u}}\,\frac{1}{2}(1+X)$$

where X is the hydrogen mass fraction, as usual. Hence for our pure hydrogen star we have $\kappa_{\mathsf{H}} = \kappa_{\mathsf{es},0}$ and for the helium star we have $\kappa_{\mathsf{H}} = \frac{1}{2}\kappa_{\mathsf{es},0}$.
If we now consider

$$F_* \propto \kappa^{-1}$$

we find that the luminosity of the helium star is increased by another factor $\kappa_{\mathsf{H}}/\kappa_{\mathsf{He}} = 2$ compared to the hydrogen star, and in total the helium star is brighter by a factor

$$\frac{\kappa_{\mathsf{H}}}{\kappa_{\mathsf{He}}}\left(\frac{\mu_{\mathsf{He}}}{\mu_{\mathsf{H}}}\right)^4 = 2\left(\frac{8}{3}\right)^4 = 101.136\,.$$

Score: **8**