Astrophysics I: Stars and Stellar Evolution

AST 4001

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Stars and Stellar Evolution, Fall 2008

Agenda

1. Recap
   - Nuclear Reactions and Energy Release
   - pp Chains and CNO Cycle
2. Energy of Stars
   - The Virial Theorem
   - Energy Conservation in Stars
3. Summary
   - Summary
   - Build Your Own Star

Overview

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Summary (I)

- nuclear reactions

$$\frac{\partial}{\partial t} Y_i = \sum_{\substack{\alpha_1, \alpha_2, \ldots \\ \beta_1, \beta_2, \ldots}} \lambda_{\alpha_1 \mathbf{1} + \alpha_2 \mathbf{2} + \ldots \to \beta_1 \mathbf{1} + \beta_2 \mathbf{2} + \ldots} \frac{\beta_i - \alpha_i}{\alpha_1 ! \alpha_2 ! \ldots} Y_1^{\alpha_1} Y_2^{\alpha_2} \ldots$$

- mass excess

$$\Delta \mathcal{M}_i = m_i - A_i \mathrm{u}$$

- nuclear energy release

$$\varepsilon_{\mathrm{nuc}} = -c^2 \sum_i \Delta \mathcal{M}_i \frac{\partial}{\partial t} Y_i$$

- a cool page with nuclear data
  `http://wwwndc.jaea.go.jp/CN04/index.html`

Summary (II)

- reaction rate

$$\langle \sigma v \rangle \propto (kT)^{-2/3}\, e^{-\frac{3}{2}\left(\frac{4\pi^2 Z_1 Z_2 e^2}{h}\right)^{2/3}\left(\frac{m}{kT}\right)^{1/3}}$$

- relation between $\langle \sigma v \rangle$ and λ

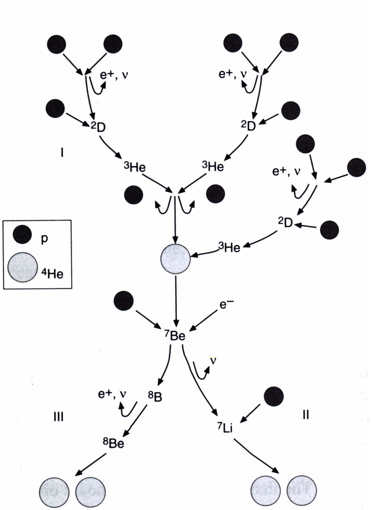
$$\lambda \propto \langle \sigma v \rangle\, \rho^m\,, \quad m = \sum \alpha_i - 1$$

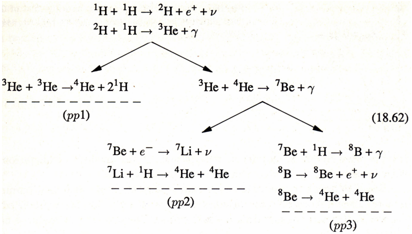
Hydrogen Burning - the two modes

- Two basic modes of hydrogen burning are distinguished
- The pp-chain in low-mass stars
- The Carbon-Nitrogen-Oxygen (CNO) cycle in high-mass stars

Hydrogen Burning - pp chains

Hydrogen burning





Energy release:

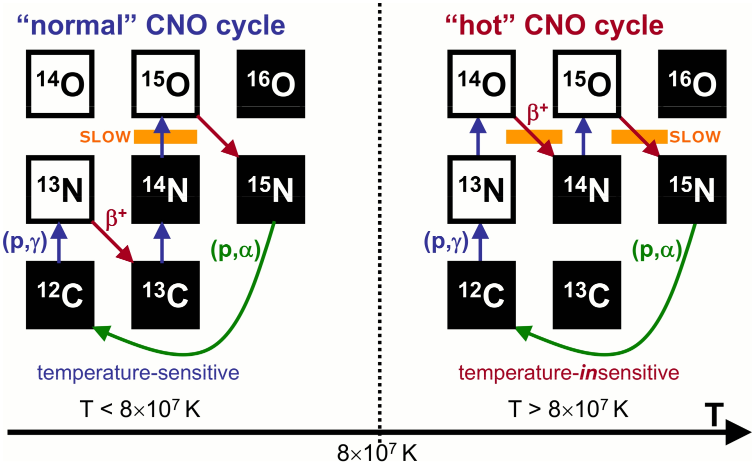
$$Q(pp1) = 26.20\,MeV$$
$$Q(pp2) = 25.67\,MeV$$
$$Q(pp3) = 19.20\,MeV$$

Reaction rate: $\langle \sigma v \rangle \propto T^4$

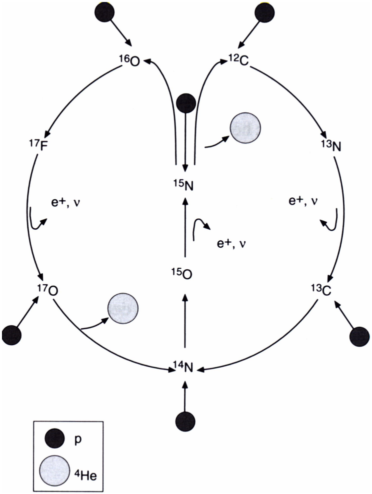
Notes on pp hydrogen burning

- All four chains fuse 4 protons to one $^4\mathrm{He}$ – and therefore have the same difference in mass excess. They have the same energy supply.
- Different Q-values (amount of energy release) due to different amounts of energy being *carried away by neutrinos*.
- With increasing temperature the dominant burning switches from pp1 to pp2 to pp3 chains.

Normal and Hot CNO Cycles



Hydrogen Burning - CNO bi-cycle

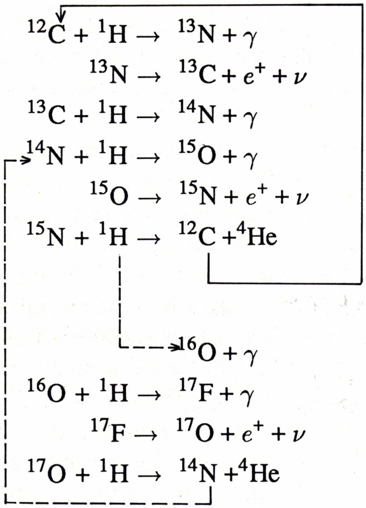


Energy release:
$Q(CNO) = 24.97 MeV$

Reaction rate: $\langle \sigma v \rangle \propto T^{16}$

Branching:
CNO-1 : CNO-2 $\sim$ 10,000 : 1

Hydrogen Burning - CNO bi-cycle



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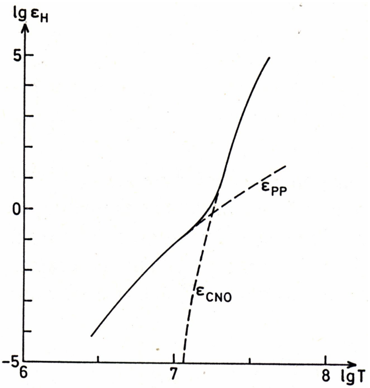
Hydrogen Burning - CNO bi-cycle

- Usually the beta-decays are fast compared to the capture reactions, (p,γ).
- $^{14}\mathrm{O}$: $\tau_{1/2} = 70$ sec
  $^{15}\mathrm{O}$: $\tau_{1/2} = 122$ sec
  $^{13}\mathrm{N}$: $\tau_{1/2} = 10\,\mathrm{min}$
  $^{17}\mathrm{F}$: $\tau_{1/2} = 64$ sec
  $^{18}\mathrm{O}$: $\tau_{1/2} = 110\,\mathrm{min}$
- $^{14}\mathrm{N}(\mathrm{p},\gamma)^{15}\mathrm{O}$ usually is the slowest “bottleneck” reaction.
- CNO cycle burning converts most CNO isotopes into $^{14}\mathrm{N}$.

Competition of Hydrogen-Burning Modes



Transition from pp-chains in low-mass stars (low T) to CNO chains in high-mass stars (high T)

Summary

- hydrogen burning can proceeds in different modes. Which mode dominates depending on temperature and the mass of the star.
- We destinguish
  - pp-chain(s)
  - CNO (bi-)cycle

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Virial Theorem

- due to hydrostatic equilibrium, internal energy of the star is connected to it gravitational binding energy
- starting from equation of hydrostatic equilibrium

$$\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4}$$

- multiply by $V = 4\pi r^3/3$ and integrate over entire star

$$\int_{P(0)}^{P(M)} V\,\mathrm{d}P = -\frac{1}{3}\int_0^M \frac{Gm}{r}\,\mathrm{d}m$$

- recall: gravitational energy of the star; define Ω

$$\Omega = -\int_0^M \frac{Gm}{r}\,\mathrm{d}m$$

Virial Theorem

- left-hand side: integration by parts

$$\int_{P(0)}^{P(M)} V\,\mathrm{d}P = [PV]_0^R - \int_{V(0)}^{V(M)} P\,\mathrm{d}V$$

- Note:
  $P(M) = 0$ (surface)
  $V(0) = 0$ (center)
  $\Rightarrow$ first term vanishes
- hence we obtain (using $\mathrm{d}m = \rho\,\mathrm{d}V$, $\mathrm{d}V = \mathrm{d}m/\rho$)

$$\Omega = -3\int_{V(0)}^{V(M)} P\,\mathrm{d}V = -3\int_0^M \frac{P}{\rho}\,\mathrm{d}m$$

Virial Theorem

- for some part of the stellar interior, $R_\mathrm{s} < R$ we may write

$$P_\mathrm{s} V_\mathrm{s} - \int_0^{M_\mathrm{s}} \frac{P}{\rho} \,\mathrm{d}m = \frac{1}{3}\Omega_\mathrm{s}$$

- where we define the gravitational binding energy of the interior as

$$\Omega_\mathrm{s} = -\int_0^{M_\mathrm{s}} \frac{Gm}{r} \,\mathrm{d}m$$

- note that this binding energy of the interior is unaffected by the outside layers!

Recap: Ideal Gas

- simple ideal gas with
  average particle mass m_{gas}
  temperature T
  density ρ
  pressure $P = \rho k_{\mathrm{B}} T / m_{\mathrm{gas}}$ (k_{B} is the Boltzmann constant)
- for gas with mean molecular weight μ with $m_{\mathrm{gas}} = \mu\,\mathrm{u}$
  we then have
  $P = \mathcal{R}\rho T/\mu$
  where $\mathcal{R} = k_{\mathrm{B}} N_{\mathrm{A}}$ is the *gas constant*
- kinetic energy per particle is $\frac{3}{2} k_{\mathrm{B}} T$
- *specific* internal energy u is

$$u = \frac{3}{2}\frac{k_{\mathrm{B}} T}{m_{\mathrm{gas}}} = \frac{3}{2}\frac{P}{\rho}$$

Virial Theorem for Ideal Gas

- total internal energy of the star

$$U = \int_0^M u \, \mathrm{d}m$$

- hence we have ($u = 3P/2\rho$)

$$\Omega = -3 \int_0^M \frac{P}{\rho} \, \mathrm{d}m = -2 \int_0^M u \, \mathrm{d}m = -2U$$

- **Virial Theorem for ideal gas**

$$U = -\frac{1}{2}\Omega$$

Virial Theorem

- generally we can write the gravitational binding energy as

$$\Omega = -\int_0^M \frac{Gm}{r}\,\mathrm{d}m = -\alpha\frac{GM^2}{R}$$

  where α is a constant of order unity depending on the internal structure of the star
- hence we can write

$$U = \frac{1}{2}\alpha\frac{GM^2}{R}$$

Virial Theorem

- we can define an average temperature of the star, $\bar{T}$, by

$$U = \int_0^M \frac{3}{2}\frac{k_\mathrm{B}T}{m_\mathrm{gas}}\,\mathrm{d}m = \frac{3}{2}\frac{k_\mathrm{B}}{m_\mathrm{gas}}\bar{T}M$$

- and using $\bar{\rho} = 3M/4\pi R^3$ we obtain

$$\bar{T} = \frac{\alpha}{3}\frac{m_\mathrm{gas}}{k_\mathrm{B}G}\frac{M}{R} = \alpha\left(\frac{4\pi}{81}\right)^{1/3}\frac{m_\mathrm{gas}}{k_\mathrm{B}G}M^{2/3}\bar{\rho}^{1/3} \propto M^{2/3}\bar{\rho}^{1/3}$$

  $\Rightarrow$ denser and more massive stars are hotter
- for $\alpha = \frac{1}{2}$ and assuming atomic hydrogen gas

$$\bar{T} \approx 4\times10^6\left(\frac{M}{\mathrm{M}_\odot}\right)\left(\frac{\mathrm{R}_\odot}{R}\right)\mathrm{K}$$

- Note that $\bar{T} \gg T_\mathrm{eff} \Rightarrow$ interior ionized plasma

Total Energy of the Star

- Integrating the energy equation

$$\frac{\partial}{\partial t}u + P\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) = q - \frac{\partial F}{\partial m}$$

over the entire star we obtain

$$\int_0^M \frac{\partial}{\partial t}u\,\mathrm{d}m + \int_0^M P\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right)\mathrm{d}m = L_{\mathrm{nuc}} - L$$

- t and m are independent variables

$$\int_0^M \frac{\partial}{\partial t}u\,\mathrm{d}m = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^M u\,\mathrm{d}m = \frac{\mathrm{d}}{\mathrm{d}t}U = \dot{U}$$

- we can write

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial V}{\partial m}\right) = \frac{\partial \dot{V}}{\partial m}$$

and $\dot{V} = 4\pi r^2 \dot{r}$

Total Energy of the Star

- integrating

$$\int_0^M P\frac{\partial \dot{V}}{\partial m}\,\mathrm{d}m = \left[P\dot{V}\right]_0^M - \int_0^M 4\pi r^2 \dot{r}\frac{\partial P}{\partial m}\,\mathrm{d}m$$

- Note:
  $P(M) = 0$ (surface)
  $\dot{V}(0) = 0$ (center)
  $\Rightarrow$ first term vanishes
- we can write

$$\dot{U} - \int_0^M 4\pi r^2 \dot{r}\frac{\partial P}{\partial m}\,\mathrm{d}m = L_{\mathrm{nuc}} - L$$

Total Energy of the Star

- equation of motion

$$\frac{\partial^2}{\partial t} r = \ddot{r} = -\frac{Gm}{r^2} - \frac{1}{\rho}\frac{\partial P}{\partial r}$$

- multiply by $\dot{r}$ and integrate over entire star

$$\int_0^M \dot{r}\ddot{r}\,\mathrm{d}m = -\int_0^M \frac{Gm}{r^2}\dot{r}\,\mathrm{d}m - \int_0^M 4\pi r^2 \dot{r}\frac{\partial P}{\partial m}\,\mathrm{d}m$$

Total Kinetic Energy of the Star

- we define the total *kinetic* energy of the star by

$$E_{\mathrm{kin}} = \int_0^M \frac{1}{2}\dot{r}^2 \,\mathrm{d}m = \int_0^M \frac{1}{2}v^2 \,\mathrm{d}m$$

- and can write

$$\int_0^M \dot{r}\ddot{r}\,\mathrm{d}m = \int_0^M \frac{\partial}{\partial t}\left(\frac{1}{2}\dot{r}^2\right)\,\mathrm{d}m = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^M \frac{1}{2}\dot{r}^2\,\mathrm{d}m = \frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{kin}} = \dot{E}_{\mathrm{kin}}$$

Total Energy of the Star

- we also have

$$-\int_0^M Gm\frac{\dot{r}}{r^2}\,\mathrm{d}m = \int_0^M Gm\frac{\partial}{\partial t}\left(\frac{1}{r}\right)\mathrm{d}m = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^M \frac{Gm}{r}\,\mathrm{d}m = -\dot{\Omega}$$

- hence we have

$$\dot{E}_{\mathrm{kin}} + \dot{\Omega} = -\int_0^M 4\pi r^2 \dot{r}\frac{\partial P}{\partial m}\,\mathrm{d}m$$

- recall

$$-\int_0^M 4\pi r^2 \dot{r}\frac{\partial P}{\partial m}\,\mathrm{d}m = L_{\mathrm{nuc}} - L - \dot{U}$$

Total Energy of the Star

- combining all terms we have

$$\dot{U} + \dot{E}_{\mathrm{kin}} + \dot{\Omega} = L_{\mathrm{nuc}} - L$$

- the total energy of the star is

$$E = U + E_{\mathrm{kin}} + \Omega$$

- therefore we have

$$\dot{E} = L_{\mathrm{nuc}} - L$$

Total Energy of the Star

Notes:

- for hydrostatic equilibrium $E_{\mathrm{kin}} = 0$
- for thermal equilibrium $\dot{E} = 0$
- hence for a hydrostatic star in thermal equilibrium U and Ω are related by virial theorem and conserved *independently*.
- for an ideal gas the total energy of the star is given by

$$E = U + \Omega = \frac{1}{2}\Omega = -U$$

- $\Rightarrow$ stars have negative heat capacity!
- $\Rightarrow$ when stars lose energy, $\dot{E} = L_{\mathrm{nuc}} - L < 0$, they contract and get hotter

Overview

Summary

- virial theorem

$$\Omega = -3 \int_{V(0)}^{V(M)} P \, \mathrm{d}V = -3 \int_0^M \frac{P}{\rho} \, \mathrm{d}m$$

- virial theorem for ideal gas

$$U = -\frac{1}{2}\Omega$$

- energy conservation

$$\dot{U} + \dot{E}_{\mathrm{kin}} + \dot{\Omega} = L_{\mathrm{nuc}} - L$$

- stars have negative heat capacity!

Stellar Evolution Project

- Bill Paxton's **EZ Stellar Evolution** code
  `http://www.kitp.ucsb.edu/~paxton/EZ-intro.html`
- Uses Linux `gfortran`
- g95 FORTRAN compiler can be downloaded for most platforms.
  `http://www.g95.org`