Astrophysics I: Stars and Stellar Evolution

AST 4001

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Agenda

Overview

Virial Theorem

- left-hand side: integration by parts

$$\int_{P(0)}^{P(M)} V\,\mathrm{d}P = [PV]_0^R - \int_{V(0)}^{V(M)} P\,\mathrm{d}V$$

- Note:
  $P(M) = 0$ (surface)
  $V(0) = 0$ (center)
  $\Rightarrow$ first term vanishes
- hence we obtain (using $\mathrm{d}m = \rho\,\mathrm{d}V$, $\mathrm{d}V = \mathrm{d}m/\rho$)

$$\Omega = -3\int_{V(0)}^{V(M)} P\,\mathrm{d}V = -3\int_0^M \frac{P}{\rho}\,\mathrm{d}m$$

Virial Theorem for Ideal Gas

- total internal energy of the star

$$U = \int_0^M u \, \mathrm{d}m$$

- hence we have ($u = 3P/2\rho$)

$$\Omega = -3 \int_0^M \frac{P}{\rho} \mathrm{d}m = -2 \int_0^M u \, \mathrm{d}m = -2U$$

- **Virial Theorem for ideal gas**

$$U = -\frac{1}{2}\Omega$$

Virial Theorem

- generally we can write the gravitational binding energy as

$$\Omega = -\int_0^M \frac{Gm}{r}\,\mathrm{d}m = -\alpha\frac{GM^2}{R}$$

  where α is a constant of order unity depending on the internal structure of the star
- hence we can write

$$U = \frac{1}{2}\alpha\frac{GM^2}{R}$$

Virial Theorem

- we can define an average temperature of the star, $\bar{T}$, by

$$U = \int_0^M \frac{3}{2}\frac{k_\mathrm{B}T}{m_\mathrm{gas}}\,\mathrm{d}m = \frac{3}{2}\frac{k_\mathrm{B}}{m_\mathrm{gas}}\bar{T}M$$

- and using $\bar{\rho} = 3M/4\pi R^3$ we obtain

$$\bar{T} = \frac{\alpha}{3}\frac{m_\mathrm{gas}}{k_\mathrm{B}G}\frac{M}{R} = \alpha\left(\frac{4\pi}{81}\right)^{1/3}\frac{m_\mathrm{gas}}{k_\mathrm{B}G}M^{2/3}\bar{\rho}^{1/3} \propto M^{2/3}\bar{\rho}^{1/3}$$

$\Rightarrow$ denser and more massive stars are hotter

- for $\alpha = \frac{1}{2}$ and assuming atomic hydrogen gas

$$\bar{T} \approx 4\times10^6\left(\frac{M}{\mathrm{M}_\odot}\right)\left(\frac{\mathrm{R}_\odot}{R}\right)\mathrm{K}$$

- Note that $\bar{T} \gg T_\mathrm{eff} \Rightarrow$ interior ionized plasma

Total Energy of the Star

- combining all terms we have

$$\dot{U} + \dot{E}_{\mathrm{kin}} + \dot{\Omega} = L_{\mathrm{nuc}} - L$$

- the total energy of the star is

$$E = U + E_{\mathrm{kin}} + \Omega$$

- therefore we have

$$\dot{E} = L_{\mathrm{nuc}} - L$$

Total Energy of the Star

Notes:

- for hydrostatic equilibrium $E_{\text{kin}} = 0$
- for thermal equilibrium $\dot{E} = 0$
- hence for a hydrostatic star in thermal equilibrium U and Ω are related by virial theorem and conserved *independently*.
- for an ideal gas the total energy of the star is given by

$$E = U + \Omega = \frac{1}{2}\Omega = -U$$

- $\Rightarrow$ stars have negative heat capacity!
- $\Rightarrow$ when stars lose energy, $\dot{E} = L_{\text{nuc}} - L < 0$, they contract and get hotter

Summary

- virial theorem

$$\Omega = -3 \int_{V(0)}^{V(M)} P \, \mathrm{d}V = -3 \int_0^M \frac{P}{\rho} \, \mathrm{d}m$$

- virial theorem for ideal gas

$$U = -\frac{1}{2}\Omega$$

- energy conservation

$$\dot{U} + \dot{E}_{\mathrm{kin}} + \dot{\Omega} = L_{\mathrm{nuc}} - L$$

- stars have negative heat capacity!

Extra Notes

- In the energy equation

$$\frac{\partial}{\partial t}u + P\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) = q - \frac{\partial F}{\partial m}$$

$$q = \varepsilon_{\mathrm{nuc}} - \varepsilon_{\nu}$$

more general allows to define neutrino luminosity:

$$L_{\nu} = \int_0^M \varepsilon_{\nu}\,\mathrm{d}m$$

- We should distinguish
  - neutrinos from *weak decays*
    (hydrogen burning, about 7 % of L_{nuc})
  - *thermal* neutrinos made in hot plasma and can cool the star without the presence of nuclear reactions.

Quiz

- How would you write the energy conservation equation including L_ν?
- Any other definition that would need to be changed? And how?
- How would you modify homework problem 1(a) to account for L_ν?

Answers

- How would you write the energy conservation equation including L_ν?
  $\dot{U} + \dot{E}_{\text{kin}} + \dot{\Omega} = L_{\text{nuc}} - L - L_\nu$
- Any other definition that would need to be changed? And how?
  $L_{\text{nuc}} = \int_0^M \varepsilon_{\text{nuc}}\,\mathrm{d}m$
- How would you modify homework problem 1(a) to account for L_ν?
  Add neutrino luminosity to solar luminosity.

Overview

General "Definition" of Time Scales

- We compare the ratio of quantity x to it rate of change,

$$\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t} x$$

  to define a time scale

$$\tau_x = \frac{x}{\dot{x}} = x \Bigg/ \frac{\mathrm{d}}{\mathrm{d}t} x = \left(\frac{\mathrm{d} \ln x}{\mathrm{d}t} \right)^{-1} = \frac{\mathrm{d}t}{\mathrm{d} \ln x}$$

- Can be defined locally, parts of the star, or for integral quantities like total energy of the star
- the three most important time scales governing the evolution of stars in their different phases are
  - dynamical time scale
  - thermal time scale
  - nuclear time scale

Dynamical Time Scale

- Consider change of radius R.
- at free fall the escape velocity at the surface is characteristic velocity

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$$

- using average density $\bar{\rho} = 3M/4\pi R^3$ we can write

$$\tau_{\text{dyn}} \approx \frac{R}{v_{\text{esc}}} = \sqrt{\frac{R^3}{2GM}} = \sqrt{\frac{3}{8\pi}}\frac{1}{\sqrt{G\bar{\rho}}} \approx \frac{1}{\sqrt{G\bar{\rho}}}$$

- an alternative definition comes from

$$-\frac{\partial^2}{\partial t^2}r\Big|_{\text{surface}} \approx R/\tau_{\text{dyn}}^2 \quad \Rightarrow \quad \tau_{\text{dyn}} \approx \sqrt{\frac{R}{g}} = \sqrt{\frac{R^3}{GM}}$$

Dynamical Time Scale

Notes:

- the dynamical time scale gives the time scale on which the star restores hydrostatic equilibrium
- sometimes we can, however, observe changes in stars on the dynamic time scale
- for example, for many kinds of stellar oscillations the dynamic time scale is the characteristic time scale.
- For the sun we roughly have

$$\tau_{\rm dyn} = 1000\,{\rm s} \times \left(\frac{R}{{\rm R}_\odot}\right)^{3/2} \left(\frac{{\rm M}_\odot}{M}\right)^{1/2}$$

Thermal Time Scale

- Consider change of internal energy U of the stars
- This is related, by the virial theorem, to the gravitational binding energy Ω of the star
- we may hence approximate

$$U \approx \alpha \frac{GM^2}{R}$$

- Using L as the time scale on which heat is radiated away, we may estimate

$$\tau_{\text{thermal}} = \frac{U}{L} \approx \alpha \frac{GM^2}{RL}$$

Thermal Time Scale

Notes:

- using a characteristic value of $\alpha = \frac{1}{2}$ we obtain the usual definition of the *Kelvin-Helmholtz time scale*

$$\tau_{\mathrm{KH}} = \frac{GM^2}{2RL}$$

- Recall: For the sun we obtain

$$\tau_{\mathrm{KH}} = 5{\times}10^{14}\,\mathrm{s} \times \left(\frac{M}{\mathrm{M}_\odot}\right)^2 \left(\frac{\mathrm{R}_\odot}{R}\right) \left(\frac{\mathrm{L}_\odot}{L}\right)$$

Nuclear Time Scale

- Consider the energy provided by nuclear burning

$$L_{\text{nuc}} = \int_0^M \varepsilon_{\text{nuc}}\,\mathrm{d}m \approx \bar{\epsilon} M c^2 / \tau_{\text{nuc}}$$

  where the average fraction of rest mass released, ϵ is of order a few 10^{-3}
- and compare with luminosity $L \approx L_{\text{nuc}}$

$$\tau_{\text{nuc}} \approx \frac{\epsilon M c^2}{L}$$

- and numerically

$$\tau_{\text{nuc}} = \epsilon\, 4.5 \times 10^{20}\,\text{s} \times \left(\frac{M}{\mathrm{M}_\odot}\right)\left(\frac{\mathrm{L}_\odot}{L}\right)$$

Summary Time Scales

- Summarizing we have

$$\tau_{\mathrm{dyn}} \ll \tau_{\mathrm{KH}} \ll \tau_{\mathrm{nuc}}$$

- stars restore hydrostatic equilibrium on τ_{dyn}
- stars restore thermal equilibrium on τ_{KH}
  This is the time scale on which they evolve in the absence of nuclear energy supply
- hydrostatic stars burning nuclear fuel evolve on timescale τ_{nuc}

Quiz

How would you define a thermal time scale when $L_\nu \gg L$?

Answer

How would you define a thermal time scale when $L_\nu \gg L$?

$$\tau_{\mathrm{KH},\nu} = \frac{GM^2}{2RL_\nu}$$

General Equation of State

- Generally we can write the equation of state as a unique function

$$P = P(\rho, T, \mathbf{X})$$

- Obviously, this could also be solved, for different applications, as

$$\rho = \rho(P, T, \mathbf{X}) \quad \text{or} \quad T = T(P, \rho, \mathbf{X})$$

- so far we have only considered the case of *ideal gas*.
- in stars density can become so high that gas particles interact, temperature is so high that atoms are ionized and that radiation contributes to the pressure

Coulomb Interaction

- For a star of average density $\bar{\rho}$ and particle mass of $\mu\,\mathrm{u}$ we have a mean distance between particles of

$$d = \left(\frac{\mu\,\mathrm{u}}{\bar{\rho}}\right)^{1/3} = \left(\frac{4\pi\mu\,\mathrm{u}}{3M}\right)^{1/3} R$$

- we may estimate the mean Coulomb Energy by

$$\epsilon_{\mathrm{Coulomb}} = \frac{Z^2 e^2}{d}$$

- we may compare to the characteristic energy of the particles in the stars $k_{\mathrm{B}}\bar{T}$ where $\bar{T} \approx \alpha\mu\,\mathrm{u}GM/3k_{\mathrm{B}}R$:

$$\frac{\epsilon_{\mathrm{Coulomb}}}{k_{\mathrm{B}}\bar{T}} \sim \frac{3Z^2 e^2}{\mu^{4/3}\,\mathrm{u}^{4/3}\alpha G M^{2/3}}$$

Coulomb Interaction

Notes:

- For the sun
$$\Gamma = \frac{\epsilon_{\mathrm{Coulomb}}}{k_{\mathrm{B}}\bar{T}} \sim 0.01$$
this Γ is often call the *Coulomb Parameter*
- for $M \lesssim 10^{-3}\,\mathrm{M}_{\odot}$, Γ becomes of order unity
this is the mass scale of planets
- typically for
$\Gamma \approx 5$: transition from gas to liquid
$\Gamma \approx 175$: transition from liquid to solid

Overview

Summary of Time Scales

- dynamic time scale

$$\tau_{\text{dyn}} \approx \frac{1}{\sqrt{G\bar{\rho}}}$$

- Kelvin-Helmholtz (thermal) time scale

$$\tau_{\text{thermal}} \approx \tau_{\text{KH}} = \frac{GM^2}{2RL}$$

- nuclear time scale

$$\tau_{\text{nuc}} \approx \frac{\epsilon Mc^2}{L}\,, \qquad \epsilon \sim 10^{-3}$$

- comparison of time scales

$$\tau_{\text{dyn}} \ll \tau_{\text{KH}} \ll \tau_{\text{nuc}}$$

Stellar Evolution Project

- Bill Paxton's **EZ Stellar Evolution** code
  `http://www.kitp.ucsb.edu/~paxton/EZ-intro.html`
- Uses Linux `gfortran`
- g95 FORTRAN compiler can be downloaded for most platforms.
  `http://www.g95.org`