Astrophysics I: Stars and Stellar Evolution
AST 4001

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Stars and Stellar Evolution, Fall 2008

Agenda

1. Recap
   - Corrections
   - Summary of Time Scales
2. Equation of State
   - Gas Pressure
   - Degenerate Gas Pressure
   - Radiation Pressure
3. Summary
   - Summary
   - Build Your Own Star

Overview

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Dynamical Time Scale

- Consider change of radius R.
- at free fall the escape velocity at the surface is characteristic velocity

$$v_{\mathrm{esc}} = \sqrt{\frac{2GM}{R}}$$

- using average density $\bar{\rho} = 3M/4\pi R^3$ we can write

$$\tau_{\mathrm{dyn}} \approx \frac{R}{v_{\mathrm{esc}}} = \sqrt{\frac{R^3}{2GM}} = \sqrt{\frac{3}{8\pi}}\frac{1}{\sqrt{G\bar{\rho}}} \approx \frac{1}{\sqrt{G\bar{\rho}}}$$

- an alternative definition comes from

$$-\frac{\partial^2}{\partial t^2} r\Big|_{\mathrm{surface}} \approx R/\tau_{\mathrm{dyn}}^2 \quad \Rightarrow \quad \tau_{\mathrm{dyn}} \approx \sqrt{\frac{R}{g}} = \sqrt{\frac{R^3}{GM}}$$

Summary of Time Scales

- dynamic time scale

$$\tau_{\rm dyn} \approx \frac{1}{\sqrt{G\bar{\rho}}}$$

- Kelvin-Helmholtz (thermal) time scale

$$\tau_{\rm thermal} \approx \tau_{\rm KH} = \frac{GM^2}{2RL}$$

- nuclear time scale

$$\tau_{\rm nuc} \approx \frac{\epsilon Mc^2}{L}, \quad \epsilon \sim 10^{-3}$$

- comparison of time scales

$$\tau_{\rm dyn} \ll \tau_{\rm KH} \ll \tau_{\rm nuc}$$

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General Equation of State

- Generally we can write the equation of state as a unique function

$$P = P(\rho, T, \mathbf{X})$$

- Obviously, this could also be solved, for different applications, as

$$\rho = \rho(P, T, \mathbf{X}) \quad \text{or} \quad T = T(P, \rho, \mathbf{X})$$

- so far we have only considered the case of *ideal gas*.
- in stars density can become so high that gas particles interact, temperature is so high that atoms are ionized and that radiation contributes to the pressure

Coulomb Interaction

- For a star of average density $\bar{\rho}$ and particle mass of $\mu\,\mathrm{u}$ we have a mean distance between particles of

$$d = \left(\frac{\mu\,\mathrm{u}}{\bar{\rho}}\right)^{1/3} = \left(\frac{4\pi\mu\,\mathrm{u}}{3M}\right)^{1/3} R$$

- we may estimate the mean Coulomb Energy by

$$\epsilon_{\mathrm{Coulomb}} = \frac{Z^2 e^2}{d}$$

- we may compare to the characteristic energy of the particles in the stars $k_{\mathrm{B}}\bar{T}$ where $\bar{T} \approx \alpha\mu\,\mathrm{u}GM/3k_{\mathrm{B}}R$:

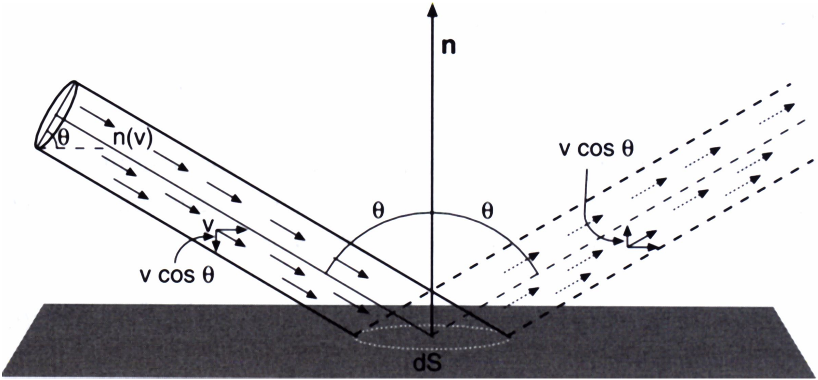
$$\frac{\epsilon_{\mathrm{Coulomb}}}{k_{\mathrm{B}}\bar{T}} \sim \frac{3Z^2 e^2}{\mu^{4/3}\,\mathrm{u}^{4/3}\alpha GM^{2/3}}$$

Coulomb Interaction

Notes:

- For the sun
$$\Gamma = \frac{\epsilon_{\mathrm{Coulomb}}}{k_{\mathrm{B}}\bar{T}} \sim 0.01$$
this Γ is often call the *Coulomb Parameter*
- for $M \lesssim 10^{-3}\,\mathrm{M}_\odot$, Γ becomes of order unity
this is the mass scale of planets
- typically for
$\Gamma \approx 5$: transition from gas to liquid
$\Gamma \approx 175$: transition from liquid to solid

Definition of Pressure from First Principles



Define pressure by momentum transfer per unit area per unit time by gas "particles". Assume particles have velocity v and hit the surface area dS at an angle Θ.

Definition of Pressure

We use

- $n(p)\,\mathrm{d}p$ is number of particles per unit volume in momentum interval $(p, p + \mathrm{d}p)$
- assume elastic reflection of particle
- momentum transferred in collision is given by twice momentum change, the momentum component perpendicular to the surface normal

$$\Delta p = 2p\cos\Theta$$

- assume isotropic particle distribution.
  We define $n(\Theta, p)\,\mathrm{d}\Theta\,\mathrm{d}p$ as the number of particles in a cone $(\Theta, \Theta + \mathrm{d}\Theta)$ and with momenta $(p, p + \mathrm{d}p)$

Definition of Pressure

- in isotropic distribution number of particles per unit solid angle $\mathrm{d}\omega$ is proportional to the solid angle itself

$$\frac{n(\Theta,p)\,\mathrm{d}\Theta\,\mathrm{d}p}{n(p)\,\mathrm{d}p}=\frac{\mathrm{d}\omega}{4\pi}=\frac{2\pi\sin\Theta\,\mathrm{d}\Theta}{4\pi}=\frac{1}{2}\sin\Theta\,\mathrm{d}\Theta$$

- the number of particles that strike surface area $\mathrm{d}S$ within a time interval δt is then $n(p,\Theta)v\delta t\,\mathrm{d}S\cos\Theta$ and they transfer a momentum $\Delta p=2p\cos\Theta$.
- The momentum transfer per incident angle Θ is then

$$\delta p_\Theta = n(\Theta,p)\,2pv\cos^2\Theta\,\mathrm{d}\Theta\,\mathrm{d}p\,\delta t\,\mathrm{d}S$$

$$= n(p)\,pv\cos^2\Theta\;\sin\Theta\,\mathrm{d}\Theta\,\mathrm{d}p\,\delta t\,\mathrm{d}S$$

Definition of Pressure Integral

- The pressure, defined as the integral of δp_Θ over all $\Theta = (0, \pi/2)$ and all momenta $p = (0, \infty)$ per unit time δt and per surface area dS is then

$$P = \int_0^\infty \int_0^{\pi/2} n(p)\, pv \cos^2\Theta\ \sin\Theta\, \mathrm{d}\Theta\, \mathrm{d}p$$

$$P = \int_0^{\pi/2} \cos^2\Theta\ \sin\Theta\, \mathrm{d}\Theta \int_0^{pi/2} v\, p\, n(p)\ \mathrm{d}p$$

- That is, we obtain the *pressure integral*

$$P = \frac{1}{3} \int_0^\infty v\, p\, n(p)\, \mathrm{d}p$$

Definition of Pressure Contributions

- Generally we have different contributions to the total pressure:

$$P = P_\mathrm{I} + P_\mathrm{e} + P_\mathrm{rad} = P_\mathrm{gas} + P_\mathrm{rad}$$

- one often defines

$$\beta = P_\mathrm{gas}/P$$

  and hence can write

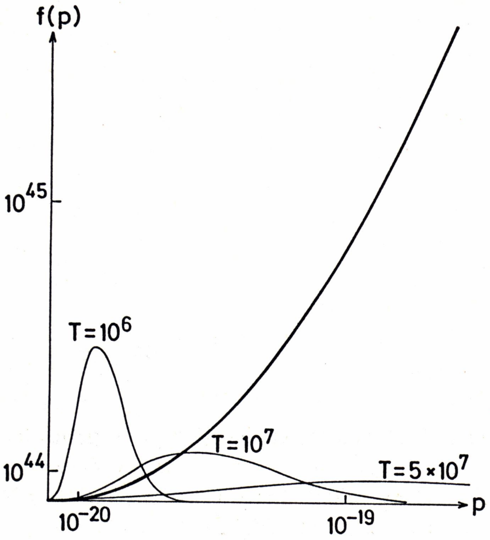
$$P_\mathrm{gas} = \beta P$$

  and

$$P_\mathrm{rad} = (1-\beta)P$$

- Obviously, $0 < \beta < 1$:
  - for $\beta \ll 1$ the pressure is called *radiation-dominated*
  - for $\beta \sim 1$ the pressure is called *gas-dominated*

Ion Pressure



Bolzman distribution at different temperatures

Ion Pressure

- Inside stars the ions usually follow, in very good approximation, the Maxwell-Boltzmann distribution

$$n(p)\,\mathrm{d}p = \frac{4\pi n_\mathrm{I} p^2\,\mathrm{d}p}{\sqrt{2\pi m_\mathrm{I} k_\mathrm{B} T}^3} e^{-\frac{p^2}{2 m_\mathrm{I} k_\mathrm{B} T}}$$

- and the resulting pressure is given by the usual

$$P_\mathrm{I} = n_\mathrm{I} k_\mathrm{B} T$$

- recall that the total number density of ions is given by adding up the contributions from all species i:

$$n_\mathrm{I} = \sum_i n_i = \sum_i \frac{\rho}{\mathrm{u}} Y_i$$

where $Y_i = X_i / A_i$ is the mole fraction of species i.

Ion Pressure

- with the definition of the *mean molecular weight* of the ions

$$\frac{1}{\mu_\mathrm{I}} = \sum_i Y_i$$

and can write

$$n_\mathrm{I} = \frac{\rho}{\mu_\mathrm{I}\mathrm{u}}$$

- for a stellar gas composed mostly of hydrogen and helium, we may write

$$\frac{1}{\mu_\mathrm{I}} \approx X + \frac{1}{4}Y + (1 - X - Y)\frac{1}{\langle A\rangle_\mathrm{metals}}$$

- for the sun we roughly have $\langle A\rangle_\mathrm{metals} \approx 20$ and hence $\mu_\mathrm{I} = 1.29$

Ion Pressure

- Using the definition of the gas constant

$$\mathcal{R} = k_\mathrm{B} N_\mathrm{A}\,, \quad (N_\mathrm{A}\mathrm{u} = 1\mathrm{g})$$

we may write the ion pressure in the usual form

$$P_\mathrm{I} = \frac{\mathcal{R}T\rho}{\mu_\mathrm{I}}$$

Ideal Electron Gas

- For an ideal (non-degenerate) electron gas we have similar to the ions

$$P_{\mathrm{e}} = n_{\mathrm{e}} k_{\mathrm{B}} T$$

where n_{e} is the number for "free" electrons.

- to good approximation, for most of the interior of most stars, the gas can be assumed fully ionized
- the number density of electrons is then given by

$$n_{\mathrm{e}} = \sum_i Z_i n_i = \frac{\rho}{\mathrm{u}} \sum_i Z_i Y_i$$

we define $\frac{1}{\mu_{\mathrm{e}}} = \sum_i Z_i Y_i$ and obtain $n_{\mathrm{e}} = \frac{\rho}{\mu_{\mathrm{e}} \mathrm{u}}$

Ideal Electron Gas Pressure

- for a stellar gas composed mostly of hydrogen and helium, we may write

$$\frac{1}{\mu_\mathrm{e}} \approx X + \frac{1}{2}Y + (1 - X - Y)\left\langle\frac{Z}{A}\right\rangle_\mathrm{metals}$$

- for most stellar compositions

$$\left\langle\frac{Z}{A}\right\rangle_\mathrm{metals} \approx \frac{1}{2}$$

and we can approximate

$$\frac{1}{\mu_\mathrm{e}} \approx \frac{1}{2}(1 + X)$$

- for the sun we have $\mu_\mathrm{e} \approx 1.17$

Total Ideal Gas Pressure

- The total pressure then is

$$P_{\mathrm{gas}} = P_{\mathrm{I}} + P_{\mathrm{e}} = \left(\frac{1}{\mu_{\mathrm{I}}} + \frac{1}{\mu_{\mathrm{e}}}\right) \mathcal{R}\rho T = \frac{\mathcal{R}\rho T}{\mu}$$

- where we defined the mean molecular weight of the gas

$$\frac{1}{\mu} = \frac{1}{\mu_{\mathrm{I}}} + \frac{1}{\mu_{\mathrm{e}}}$$

- for the sun we have $\mu \approx 0.61$

Degenerate Electron Pressure

- electrons much lighter than ions ($m_e \ll m_I$) less momentum for a given energy $k_B T \Rightarrow$ become degenerate first
- Heisenberg's uncertainty principle

$$\Delta^3 x \Delta^3 p \geq h^3$$

plus two spin orientations

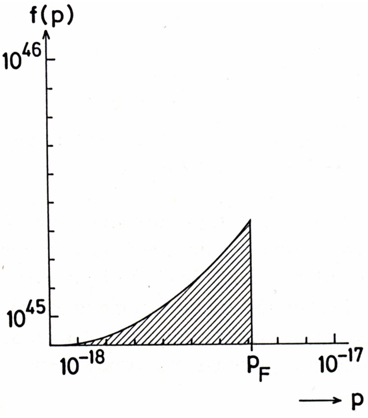
- for a completely degenerate isotropic electron gas we have the distribution function

$$n_e(p)\,\mathrm{d}p = \frac{2}{\Delta^3 x} = \frac{2}{h^3} 4\pi p^2\,\mathrm{d}p$$

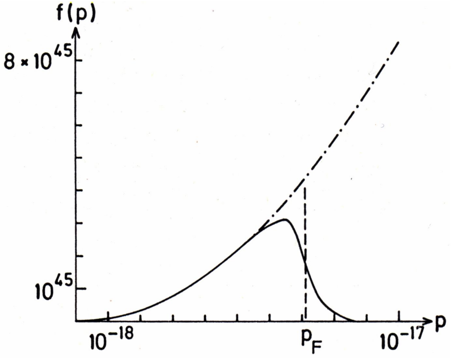
for $p < p_F$ and $n_e(p)\,\mathrm{d}p = 0$ otherwise

Degenerate Electron Gas

fully degenerate



partially degenerate



Degenerate Electron Pressure

- The maximum ("Fermi") momentum is obtained by integrating

$$n_\mathrm{e} = \int_0^{p_\mathrm{F}} n_\mathrm{e}(p)\,\mathrm{d}p$$

to obtain

$$p_\mathrm{F} = \left(\frac{3h^3 n_\mathrm{e}}{8\pi}\right)^{1/3}$$

- the pressure integral than gives

$$P_\mathrm{e,deg} = \frac{8\pi {p_\mathrm{F}}^5}{14 m_\mathrm{e} h^3} = \frac{h^2}{20 m_\mathrm{e} \mathrm{u}^{5/3}} \left(\frac{3}{\pi}\right)^{2/3} \left(\frac{\rho}{\mu_\mathrm{e}}\right)^{5/3} = K_1 \rho^{5/3} \propto \rho^{5/3}$$

using $n_\mathrm{e} = \rho/\mu_\mathrm{e}\mathrm{u}$, and $K_1 \approx 10^{13} \frac{\mathrm{dyn\,cm^{-2}}}{(\mathrm{g cm^{-3}})^{5/3}}$.

Degenerate Relativistic Pressure

- generally we have

$$v = \frac{p}{m_e} \Bigg/ \sqrt{1 + \frac{p^2}{{m_e}^2 c^2}}$$

but for $p \gg m_e c$ we have $v \to c$ and we replace v by c in the pressure integral and after integration obtain

$$P_{\mathrm{e,rel-deg}} = \frac{hc}{8\mathrm{u}^{4/3}} \left(\frac{3}{\pi}\right)^{1/3} \left(\frac{\rho}{\mu_e}\right)^{4/3} = K_2 \rho^{4/3} \propto \rho^{4/3}$$

and $K_2 \approx 1.24 \times 10^{15} \frac{\mathrm{dyn\,cm}^{-2}}{(\mathrm{gcm}^{-3})^{4/3}}$.

- Note that for the degenerate electron gas the pressure is insensitive to temperature.

Radiation Pressure

- From the Planck spectrum

$$n(\nu)\mathrm{d}\nu = \frac{8\pi\nu^2}{c^3}\frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}}-1}\mathrm{d}\nu$$

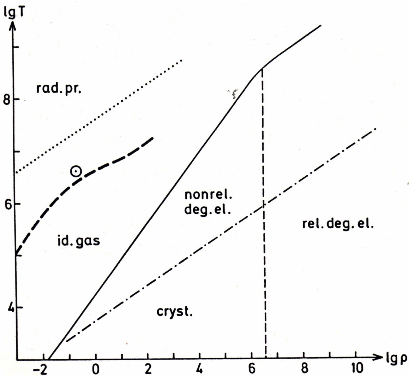
we obtain

$$P_\mathrm{rad} = \frac{1}{3}\int_0^\infty c\frac{h\nu}{c}\,n(\nu)\,\mathrm{d}\nu = \frac{1}{3}aT^4$$

with the radiation constant

$$a = \frac{8\pi^5 {k_\mathrm{B}}^4}{15c^3h^3} = \frac{4\sigma}{c}$$

Regimes of the EOS



Different regimes of the equation of state as a function of T and ρ.

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Summary of Pressure Contributions

- Pressure integral $P = \frac{1}{3}\int_0^\infty v\,p\,n(p)\,\mathrm{d}p$
- $P = P_\mathrm{I} + P_\mathrm{e} + P_\mathrm{rad} = P_\mathrm{gas} + P_\mathrm{rad}$
  define $\beta = P_\mathrm{gas}/P \Rightarrow P_\mathrm{gas} = \beta P$, $P_\mathrm{rad} = (1-\beta)P$
- gas pressure
  $P_\mathrm{gas} = \mathcal{R}\rho\frac{T}{\mu}$
- degenerate electron pressure
  $$P_\mathrm{e,deg} = \frac{h^2}{20 m_\mathrm{e} \mathrm{u}^{5/3}}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{\rho}{\mu_\mathrm{e}}\right)^{5/3}$$
- relativistic degenerate electron pressure
  $$P_\mathrm{e,rel-deg} = \frac{hc}{8\mathrm{u}^{4/3}}\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{\rho}{\mu_\mathrm{e}}\right)^{4/3}$$
- radiation pressure
  $P_\mathrm{rad} = \frac{1}{3}\int_0^\infty c\frac{h\nu}{c}\,n(\nu)\,\mathrm{d}\nu = \frac{1}{3}aT^4$

Stellar Evolution Project

- Bill Paxton's **EZ Stellar Evolution** code
  `http://www.kitp.ucsb.edu/~paxton/EZ-intro.html`
- Uses Linux `gfortran`
- g95 FORTRAN compiler can be downloaded for most platforms.
  `http://www.g95.org`