Astrophysics I: Stars and Stellar Evolution
AST 4001

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Stars and Stellar Evolution, Fall 2008

Agenda

1. Recap
   - Pressure Contributions
   - Regimes of the EOS
2. Thermodynamics of Stellar Gas
   - Energy Density
   - Adiabatic Index
   - Saha Equation
3. Summary
   - Summary
   - Build Your Own Star

Overview

Summary of Pressure Contributions

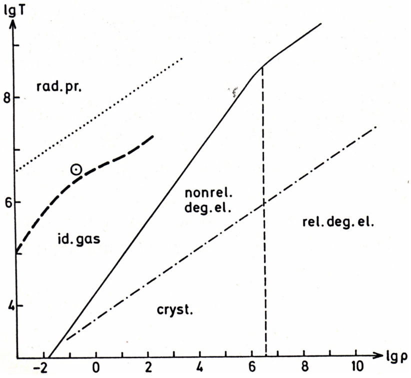
- Pressure integral $P = \frac{1}{3}\int_0^\infty v\,p\,n(p)\,\mathrm{d}p$, $v = \frac{p}{m_\mathrm{e}}\Big/\sqrt{1+\frac{p^2}{{m_\mathrm{e}}^2c^2}}$
- $P = P_\mathrm{I} + P_\mathrm{e} + P_\mathrm{rad} = P_\mathrm{gas} + P_\mathrm{rad}$
  define $\beta = P_\mathrm{gas}/P \Rightarrow P_\mathrm{gas} = \beta P$, $P_\mathrm{rad} = (1-\beta)P$
- gas pressure
  $P_\mathrm{gas} = \mathcal{R}\rho\frac{T}{\mu}$
- degenerate electron pressure
  $$P_\mathrm{e,deg} = \frac{h^2}{20 m_\mathrm{e} \mathrm{u}^{5/3}}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{\rho}{\mu_\mathrm{e}}\right)^{5/3}$$
- relativistic degenerate electron pressure
  $$P_\mathrm{e,rel-deg} = \frac{hc}{8\mathrm{u}^{4/3}}\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{\rho}{\mu_\mathrm{e}}\right)^{4/3}$$
- radiation pressure
  $P_\mathrm{rad} = \frac{1}{3}\int_0^\infty c\frac{h\nu}{c}\,n(\nu)\,\mathrm{d}\nu = \frac{1}{3}aT^4$

Notes on Pressure Contributions

Notes:

- In regime of (relativistic) degenerate electron gas $P_{\text{gas}} = P_{\text{I}} + P_{\text{e,deg}}$ $(P = P_{\text{I}} + P_{\text{e,rel-deg}})$
- (relativistic) electron degeneracy pressure is proportional to powers of $\left(\frac{\rho}{\mu_{\text{e}}}\right)$
  in particular it does not depend on the ion mass - the kind of nuclei - as long as there is the same ratio of neutrons to protons
- gas pressure depends on $\frac{T\rho}{\mu}$
- radiation pressure depends on T^4

Regimes of the EOS



Different regimes of the equation of state as a function of T and ρ.

Overview

Energy Density of Stellar Gas

- Generally we can write the *specific* energy density as

$$u = \frac{1}{\rho} \int_0^\infty n(p) \epsilon(p) \, \mathrm{d}p$$

where we use the (relativistic) energy per particle with mass m_{gas} of

$$\epsilon(p) = m_{\mathrm{gas}} c^2 \left[\left(1 + \frac{p^2}{m_{\mathrm{gas}}^2 c^2} \right)^{1/2} - 1 \right]$$

- for $p \ll m_{\mathrm{gas}} c$ we recover the classical limit $\epsilon(p) = p^2 / 2 m_{\mathrm{gas}}$
- for $p \gg m_{\mathrm{gas}} c$ we obtain the fully relativistic limit $\epsilon(p) = p\, c$

Quiz

Derive the limits for $p \ll m_{\rm gas} c$ and $p \gg m_{\rm gas} c$ from

$$\epsilon(p) = m_{\rm gas} c^2 \left[\left(1 + \frac{p^2}{m_{\rm gas}^2 c^2} \right)^{1/2} - 1 \right]$$

together with your neighbor. Write down your solution.

Non-relativistic Energy Density

Using the Boltzmann distribution function

$$n(p) = \frac{4\pi n_\mathrm{I} p^2}{\sqrt{2\pi m_\mathrm{I} k_\mathrm{B} T}^3} e^{-\frac{p^2}{2 m_\mathrm{I} k_\mathrm{B} T}}$$

or the distribution function for a completely degenerate non-relativistic gas

$$n(p) = \begin{cases} 8\pi p^2/h^3 & \text{for} \quad p \le p_\mathrm{F} \\ 0 & \text{for} \quad p > p_\mathrm{F} \end{cases}$$

where $p_\mathrm{F} = \left(3h^3 n_\mathrm{gas}/8\pi\right)^{1/3}$, number density $n_\mathrm{gas} = \rho_\mathrm{gas}/m_\mathrm{gas}$ we obtain the usual expression

$$u_\mathrm{gas} = \frac{3}{2}\frac{P_\mathrm{gas}}{\rho}$$

Relativistic Energy Density

- for relativistic gas we obtain, after integration, accordingly

$$u_{\mathrm{gas}} = 3\frac{P_{\mathrm{gas}}}{\rho}$$

- for radiation we compute

$$u_{\mathrm{rad}} = \frac{1}{\rho}\int_0^\infty h\,\nu\,n(\nu)\,\mathrm{d}\nu = \frac{1}{\rho}\int_0^\infty h\,\nu\frac{8\pi\nu^2}{c^3}\frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}}-1}\,\mathrm{d}\nu$$

and obtain

$$u_{\mathrm{rad}} = \frac{a\,T^4}{\rho} = 3\frac{P_{\mathrm{rad}}}{\rho}$$

Adiabatic Exponent

- for “adiabatic”processes - without heat exchange we have from the first law of thermodynamics

$$\mathrm{d}q = 0 = \mathrm{d}u + P\,\mathrm{d}\left(\frac{1}{\rho}\right)$$

- in most cases we find that u is proportional to P/ρ with a constant ϕ specific to the type of gas

$$u = \phi\frac{P}{\rho}$$

- inserting we find

$$0 = \mathrm{d}\left(\phi\frac{P}{\rho}\right) + P\,\mathrm{d}\left(\frac{1}{\rho}\right) = \frac{\phi}{\rho}\,\mathrm{d}P - (\phi+1)\frac{P}{\rho^2}\,\mathrm{d}\rho$$

Adiabatic Exponent

- and then we obtain

$$\frac{\mathrm{d}P}{\mathrm{d}\rho} = \frac{\phi + 1}{\phi} \frac{P}{\rho}$$

- and define the adiabatic exponent

$$\gamma_{\mathrm{ad}} = \frac{\mathrm{d} \ln P}{\mathrm{d} \ln \rho} = \frac{\phi + 1}{\phi}$$

  and ϕ is usually called the *adiabatic exponent*.
- Hence we have

$$P = K_{\mathrm{ad}}\, \rho^{\gamma_{\mathrm{ad}}} \propto \rho^{\gamma_{\mathrm{ad}}}$$

- later we will use this relation to construct “simple” stellar models

Saha Equation

- In the outer layers of the star, our previous assumption of complete ionization may not be entirely valid
- consider a simple gas like hydrogen that can lose one electron by ionization

$$\mathrm{H} \rightleftharpoons \mathrm{H}^{+} + \mathrm{e}^{-}$$

- define
  n_0 : number density of H
  n_+ : number density of H^+
  n_{e^-} : number density of e^-
- the total particle density is then $n = n_0 + n_+ + n_{\mathrm{e}^-}$
- total density is $\rho = (n_0 + n_+)m_{\mathrm{gas}}$
- and the pressure by the (non-relativistic, non-degenerate) gas is $P = n\,k_{\mathrm{B}}T$

Saha Equation

- we define the *degree of ionization* by

$$x = \frac{n_+}{n_0 + n_+}$$

- The densities are related by the Saha Equation

$$\frac{n_+ \, n_{\mathrm{e}^-}}{n_0} = \frac{g}{h^3} (2\pi m_\mathrm{e} k_\mathrm{B} T)^{3/2} e^{-\chi/k_\mathrm{B} T}$$

  where
  g is a "statistical weight" factor
  χ is the ionization potential.

- Hence we can write the pressure as

$$P = (1 + x)(n_0 + n_+) k_\mathrm{B} T = (1 + x) \mathcal{R} \rho T$$

Saha Equation

- With this the Saha equation becomes

$$\frac{x^2}{1-x^2} = \frac{g}{h^3}\frac{(2\pi m_\mathrm{e})^{3/2}(k_\mathrm{B}T)^{5/2}}{P}\,e^{-\chi/k_\mathrm{B}T}$$

- next we have to add the ionization energy

$$\frac{\chi n_+}{\rho} = \chi\frac{n_+}{(n_0+n_+)\mathrm{u}} = \chi\frac{x}{\mathrm{u}}$$

to the total internal energy of the gas

$$u = \frac{3}{2}\frac{P}{\rho} + \frac{\chi}{\mathrm{u}}\,x$$

Saha Equation

- The ionization fraction is a function of P and ρ

$$x = x(P, \rho)$$

- differentiation gives

$$0 = \frac{3}{2}\left(\frac{1}{\rho}\right) \mathrm{d}P - \frac{3}{2}\frac{P}{\rho^2}\mathrm{d}\rho + \frac{\chi}{\mathrm{u}}\frac{\partial x}{\partial P}\mathrm{d}P + \frac{\chi}{\mathrm{u}}\frac{\partial x}{\partial \rho}\mathrm{d}P - \frac{P}{\rho^2}\mathrm{d}\rho$$

and we obtain

$$\left[\frac{3}{2} + \frac{\chi}{k_\mathrm{B}T}\left(\frac{P}{1+x}\right)\left(\frac{\partial x}{\partial P}\right)_\rho\right]\frac{\mathrm{d}P}{P} - \left[\frac{5}{2} + \frac{\chi}{k_\mathrm{B}T}\left(\frac{\rho}{1+x}\right)\left(\frac{\partial x}{\partial \rho}\right)_P\right]\frac{\mathrm{d}\rho}{\rho} = 0$$

Saha Equation

and after some lengthy manipulation one can obtain an adiabatic index

$$\gamma_{\mathrm{ad}} = \frac{5 + \left(\frac{5}{2} + \frac{\chi}{k_{\mathrm{B}} T}\right)^2 x(1-x)}{3 + \left[\frac{3}{2} + \left(\frac{3}{2} + \frac{\chi}{k_{\mathrm{B}} T}\right)^2\right] x(1-x)}$$

Notes:

- for $x = 0$ or $x = 1$ we obtain the usual value $\gamma_{\mathrm{ad}} = 5/3$
- a minimum value of γ_{ad} is obtained for $x = 0.5$
- for example, for $\chi = 10 k_{\mathrm{B}} T$ one finds $\gamma_{\mathrm{ad}} = 1.21$

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Summary of Stellar Gas

- non-relativistic gas

$$u_{\text{gas}} = \frac{3}{2}\frac{P_{\text{gas}}}{\rho}$$

- relativistic gas (ions *or* photons)

$$u_{\text{rad}} = 3\frac{P_{\text{rad}}}{\rho}$$

- adiabatic index

$$\gamma_{\text{ad}} = \frac{\mathrm{d}\ln P}{\mathrm{d}\ln\rho} = \frac{\phi+1}{\phi}$$

Stellar Evolution Project

- Bill Paxton's **EZ Stellar Evolution** code
  `http://www.kitp.ucsb.edu/~paxton/EZ-intro.html`
- Uses Linux `gfortran`
- g95 FORTRAN compiler can be downloaded for most platforms.
  `http://www.g95.org`