Astrophysics I: Stars and Stellar Evolution

AST 4001

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Stars and Stellar Evolution, Fall 2008

Overview

Computer Class

Room 575, Walter Library

Friday, 09:00-11:00

meet at fron tesk on 5th floor at 09:00.

Overview

Stellar Structure Equations

stationary terms time-dependent terms

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \tag{1}$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} - \frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} \tag{2}$$

$$\frac{\partial F}{\partial m} = \varepsilon_{\mathrm{nuc}} - \varepsilon_\nu - c_P \frac{\partial T}{\partial t} + \frac{\delta}{\rho}\frac{\partial P}{\partial t} \tag{3}$$

$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla \left[1 + \frac{r^2}{Gm}\frac{\partial^2 r}{\partial t^2}\right] \tag{4}$$

$$\frac{\partial X_i}{\partial t} = f_i(\rho, T, \mathbf{X}) \tag{5}$$

where $\mathbf{X} = \{X_1, X_2, \ldots, X_i, \ldots\}$.

Equation of State

Equation of state

$$P = \frac{\mathcal{R}}{\mu_\mathrm{I}}\rho T + P_\mathrm{e} + \frac{1}{3}aT^4$$

where P_e is the electron pressure that can be due to

- ideal electron gas,
- non-relativistic degenerate electron gas, or
- relativistic degenerate electron gas

Temperature Gradient

- The temperature gradient ∇ can be the radiative temperature gradient

$$\nabla_{\rm rad} = \left(\frac{\mathrm{d}\,\ln T}{\mathrm{d}\,\ln P}\right)_{\rm rad} = \frac{3}{16\pi acG}\frac{\kappa FP}{mT^4}$$

  or close to the adiabatic temperature gradient in convective regions

$$\nabla_{\rm ad} = \left(\frac{\partial\ln T}{\partial\ln P}\right)_{\rm ad}$$

- usually, in hydrostatic stars in thermal equilibrium ∇ is between $\nabla_{\rm ad}$ and $\nabla_{\rm rad}$:

$$\nabla_{\rm ad} \leq \nabla \leq \nabla_{\rm rad} \quad \text{or} \quad \nabla_{\rm ad} \geq \nabla \geq \nabla_{\rm rad}$$

Boundary Conditions

- In the center
  - $r = 0$
  - $m = 0$
  - $F = 0$
- at the surface
  - $r = R$
  - $m = M$
  - $F = L$
- at the surface we can also use the effective temperature

$$L = 4\pi R^2 \sigma T_{\mathrm{eff}}^4$$

Simple Model Assumptions

Assume that

- Temperature, density, and pressure decrease outward
- chemical homogeneity
- luminosity increases outward
- spherially symmetric
- neglect rotation, magnetic fields, binary star companions

Polytropic Equation of State

- the first two equations couple to the second set only indirectly through the dependence of pressure on temperature
- let us assume pressure only depends on temperature using a simple power law

$$P = K\rho^\gamma$$

- Note that this γ is *not* the adiabatic exponent, which is a property of the gas, but γ is called the *polytropic exponent* and is a property of the stellar model that takes into account temperature gradients!
- we call this a *polytropic equation of state* with a polytropic index n defined by

$$\gamma = 1 + \frac{1}{n}$$

Simple Solutions

- let us multiply the equation for pressure gradient with respect to radius,

$$\frac{\partial P}{\partial r} = -\rho \frac{Gm}{r^2}$$

by r^2/ρ and differentiate with respect to r, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -G\frac{\mathrm{d}m}{\mathrm{d}r}$$

- substituting

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}$$

in this, we obtain

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho$$

Simple Solutions for Polytrope

- let us now use the polytropic equation of state,

$$P = K\rho^{\gamma} = K\rho^{1+1/n}$$

to obtain an ordinary differential equation for density alone:

$$\frac{(n+1)K}{4\pi G n}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\rho^{(1-n)/n}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\rho$$

- the solution for $\rho(r)$ is called a *polytrope* of index n.
- it requires two boundary conditions:
  - $\rho = 0$ at the surface, $r = R$, since $P(R) = 0$
  - $\frac{\mathrm{d}P}{\mathrm{d}r} = 0$ in the center, $r = 0$, since $\frac{\mathrm{d}\rho}{\mathrm{d}r} = 0$
- the polytrope is then uniquely defined by K, n, and R
- from this we can compute other quantities, like P or m

Simple Solutions for Polytrope

- we can introduce a dimensionless variable θ with

$$0 \leq \theta \leq 1$$

by normalization to the central density:

$$\rho = \rho_\mathrm{c}\theta^n$$

- we then obtain

$$\left[\frac{(n+1)K}{4\pi G n}\right]\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left({\rho_\mathrm{c}}^{\frac{1-n}{n}}\theta^{1-n}r^2\frac{\mathrm{d}(\rho_\mathrm{c}\theta^n)}{\mathrm{d}r}\right) = -\rho_\mathrm{c}\theta^n$$

$$\left[\frac{(n+1)K}{4\pi G n {\rho_\mathrm{c}}^{\frac{n-1}{n}}}\right]\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\theta^{1-n}r^2\, n\,\theta^{n-1}\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -\theta^n$$

$$\left[\frac{(n+1)K}{4\pi G {\rho_\mathrm{c}}^{\frac{n-1}{n}}}\right]\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -\theta^n$$

Lane-Emden Equation

- Defining

$$\alpha = \sqrt{\frac{(n+1)K}{4\pi G\,\rho_c^{\frac{n-1}{n}}}}$$

and substituting $r = \alpha\xi$ we have

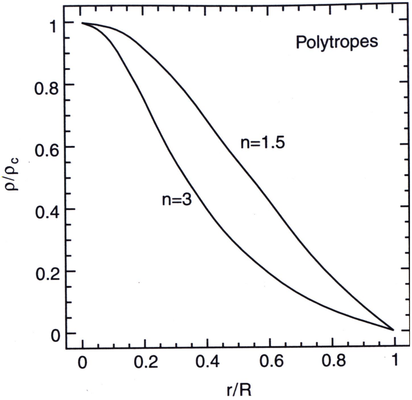
$$\alpha^2 \frac{1}{(\alpha\xi)^2} \frac{d}{d(\alpha\xi)} \left((\alpha\xi)^2 \frac{d\theta}{d(\alpha\xi)} \right) = -\theta^n$$

an obtain the **Lane-Emden Equation** – n is only parameter –

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n$$

with boundary conditions $\theta = 1$ and $\frac{d\theta}{d\xi} = 0$ at $\xi = 0$

Polytropes



- The Lane-Emden Equation can be integrated starting at $\xi = 0$.
- for $n < 5$ we find monotonically decreasing solutions
- define *radius* of star as the point where the solution of the Lane-Emden Equation drops to zero, ξ_1, we get

$$R = \alpha \xi_1$$

- $\Rightarrow$ structure of polytrope only depends on n!

Total Mass of Star

- We can integrate the polytrope starting with

$$M = \int_0^R 4\pi r^2 \rho \, \mathrm{d}r = 4\pi\alpha^3 \rho_\mathrm{c} \int_0^{\xi_1} \xi^2 \theta^n \, \mathrm{d}\xi$$

after substitution of $r = \alpha\xi$ and $\rho = \rho_\mathrm{c}\theta^n$.

- Using the Lane-Emden Equation we then substitute

$$\xi^2 \theta^n = -\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)$$

and obtain

$$M = -4\pi\alpha^3 \rho_\mathrm{c} \int_0^{\xi_1} \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) \mathrm{d}\xi = -4\pi\alpha^3 \rho_\mathrm{c} \xi_1^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi_1}$$

Definition of Polytropic Constants - Mass and Radius

- We can now define

$$M_n = -\xi_1^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi_1} > 0$$

where both ξ_1 and $\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi_1}$ are constants determined from the solution of the Lane-Emden Equation

$$M = 4\pi\alpha^3 \rho_c M_n$$

- similarly we define for a polytrope of index n

$$R_n = \xi_1 \quad \text{and obtain} \quad R = \alpha R_n$$

Polytropic Mass-Radius Relation

- In the relation $M = 4\pi\alpha^3\rho_c M_n$ let us now eliminate ρ_c from the definition of $\alpha^2 = (n+1)K \Big/ 4\pi G\, \rho_c^{\frac{n-1}{n}}$

$$M = 4\pi\alpha^3 \left(\frac{(n+1)K}{4\pi G\,\alpha^2}\right)^{\frac{n}{n-1}} M_n$$

and then eliminate α using $R = \alpha R_n$, $\alpha = R/R_n$:

$$\left(\frac{GM}{M_n}\right)^{n-1} = \frac{(4\pi)^{n-1+n}\alpha^{3n-3-2n}}{G^{n-(n-1)}}[(n+1)K]^n = \frac{\alpha^{n-3}}{4\pi G}[(n+1)K]^n$$

$$\left(\frac{GM}{M_n}\right)^{n-1} = \left(\frac{R}{R_n}\right)^{n-3} \frac{[(n+1)K]^n}{4\pi G}$$

Properties of Polytropic Mass-Radius Relation

- We now have the *Polytropic Mass-Radius Relation*:

$$\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n} = \frac{[(n+1)K]^n}{4\pi G}$$

- for $n = 3$ mass becomes independent of radius and is only determined by K:

$$M = 4\pi M_3 \left(\frac{K}{\pi G}\right)^{3/2}$$

$\Rightarrow$ there is only one possible mass that will satisfy hydrostatic equilibrium

Properties of Polytropic Mass-Radius Relation

$$\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n}=\frac{[(n+1)K]^n}{4\pi G}$$

- for $n = 1$ radius becomes independent of mass and is only determined by K:

$$R = 4\pi R_1\left(\frac{K}{2\pi G}\right)^{1/2}$$

- for $1 < n < 3$ we have $R^{3-n} \propto M^{1-n}$:
  more massive stars are denser $(3-n>0\,,\;1-n<0)$
- note, however, that n may be a function of stellar mass (more massive stars are usually less dense)

Definition of Polytropic Constants - Density

- the central density of the star is then

$$\rho_{\mathrm{c}} = -\frac{M}{4\pi\alpha^3\xi^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi_1}} = -\frac{3M}{4\pi R^3}\frac{1}{\left(\frac{3}{\xi_1}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi_1}\right)} = \bar{\rho}D_n$$

with

$$\bar{\rho} = \frac{3M}{4\pi R^3} \quad \text{and} \quad D_n = \left(\frac{3}{\xi_1}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi_1}\right)^{-1}$$

- Note that the central density, ρ_{c}, is linearly related to the average density, $\bar{\rho}$, and D_n is a constant only depending on n.

Definition of Polytropic Constants - Pressure

- From
$$P = K\rho^{\frac{n+1}{n}}$$
and replacing K from the mass-radius relation we obtain a relation for the central pressure:
$$P_\mathrm{c} = \frac{(4\pi G)^{1/n}}{n+1}\left(\frac{GM}{M_n}\right)^{\frac{n-1}{n}}\left(\frac{R}{R_n}\right)^{\frac{3-n}{n}}\rho_\mathrm{c}^{\frac{n+1}{n}} = \sqrt[3]{4\pi}B_n G M^{2/3}\rho_c^{4/3}$$
where we define a B_n that collects all the dependences on polytropic index n and only varies very slowly with n.
- hence the above relation is almost universally applicable to polytropic stars.

Polytropic Constants

Table 5.1 Polytropic constants

n	D_n	M_n	R_n	B_n
1.0	3.290	3.14	3.14	0.233
1.5	5.991	2.71	3.65	0.206
2.0	11.40	2.41	4.35	0.185
2.5	23.41	2.19	5.36	0.170
3.0	54.18	2.02	6.90	0.157
3.5	152.9	1.89	9.54	0.145

Polytropic constants for selected polytropes.

White Dwarf Mass-Radius Relation

- White dwarf stars: mass $\sim \mathrm{M}_\odot$, radius $\sim$ earth radius, cold $\Rightarrow$ Well described by (non-relativistic) degenerate equation of state with $\mu_\mathrm{e} = 2$, $P_\mathrm{e,deg} = K_1 \rho^{5/3} \Rightarrow K = K_1$ and $n = 1.5$.
- from the mass-radius relation,

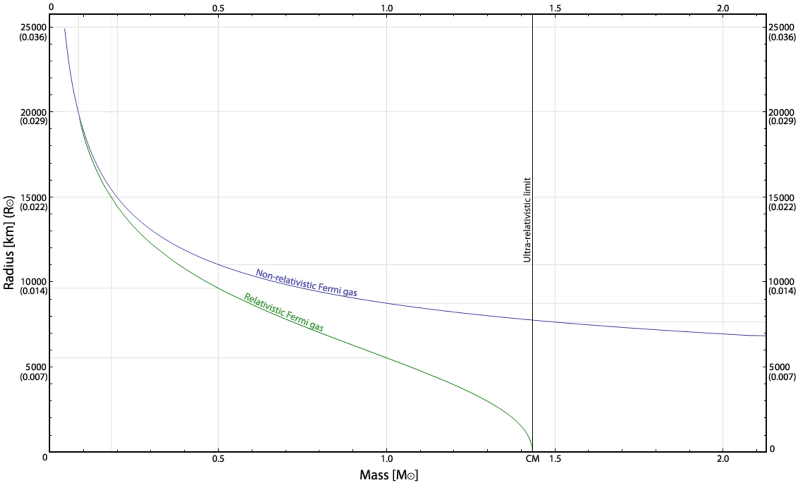
$$\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n} = \frac{[(n+1)K]^n}{4\pi G}$$

we then find

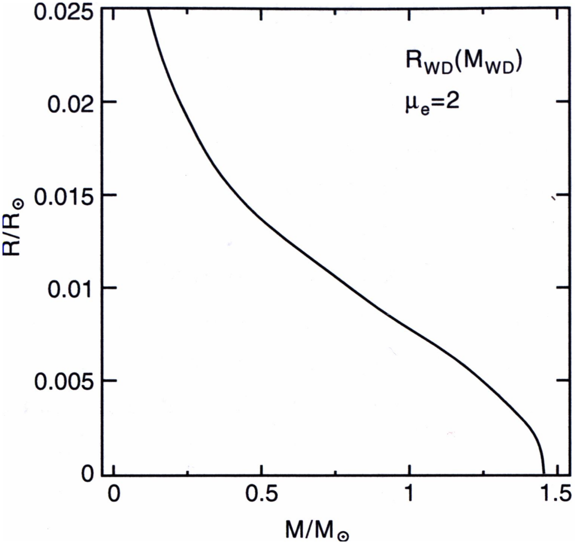
$$R \propto M^{-1/3}, \quad \bar{\rho} \propto MR^{-3} \propto M^2$$

- NOTE: for increasing mass, the radius decreases and the density increases.
- eventually the density becomes so high that we can no longer use non-relativistic degenerate equation of state.

White Dwarf Mass-Radius Relation



White Dwarf Mass-Radius Relation



- WD mass diverges for $M \to 0$
- WD mass goes to zero at Chandrasekhar mass

White Dwarf Maximum Mass

- When we use the relativistic degenerate equation of state ($\mu_\mathrm{e} = 2$),

$$P_{\mathrm{e,rel-deg}} = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\frac{1}{\mathrm{u}^{4/3}}\left(\frac{\rho}{\mu_\mathrm{e}}\right)^{4/3} = K_2\rho^{4/3}$$

  we have a polytrope with $K = K_2$ and $n = 3$.
- we recall that for $n = 3$ there is only one unique mass as solution

$$M = 4\pi M_3\left(\frac{K}{\pi G}\right)^{3/2}$$

- This determines the maximum mass of white dwarfs