Astrophysics I: Stars and Stellar Evolution

AST 4001

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Stars and Stellar Evolution, Fall 2008

Overview

1. Recap
   - Eddington (“Standard” Stellar Model)
   - Final Quiz on Eddington
2. Stellar Stability
   - Secular Thermal Stability (The Stellar Thermometer)
   - Degenerate Thermonuclear Runaway
   - Thin Shell Instability

Eddington Model

- usually star burns most fuel in center (high F/m)
- usually opacity increases outward (high κ)
- $\Rightarrow$ Eddington makes simple assumption:

$$\frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}P} = \frac{L}{4\pi c G M}\kappa\eta\,, \qquad \kappa\eta = \mathrm{constant} = \kappa_{\mathrm{surf}}$$

where κ_{surf} is the surface opacity ($\eta = 0$)

- we now have

$$\frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}P} = \frac{L}{4\pi c G M \kappa_{\mathrm{surf}}} = \mathrm{constant}$$

and obtain

$$P_{\mathrm{rad}} = \frac{L}{4\pi c G M \kappa_{\mathrm{surf}}} P$$

- $\Rightarrow$ constant ratio of gas pressure to total pressure

Eddington Model

- From

$$T = \left[\frac{3\mathcal{R}(1-\beta)}{a\mu\beta}\right]^{1/3} \rho^{1/3}$$

- and $P = \mathcal{R}T\rho/\mu\beta$ we have

$$P = K\rho^{4/3}\,, \quad K = \left[\frac{3\mathcal{R}^4(1-\beta)}{a\mu^4\beta^4}\right]^{1/3}$$

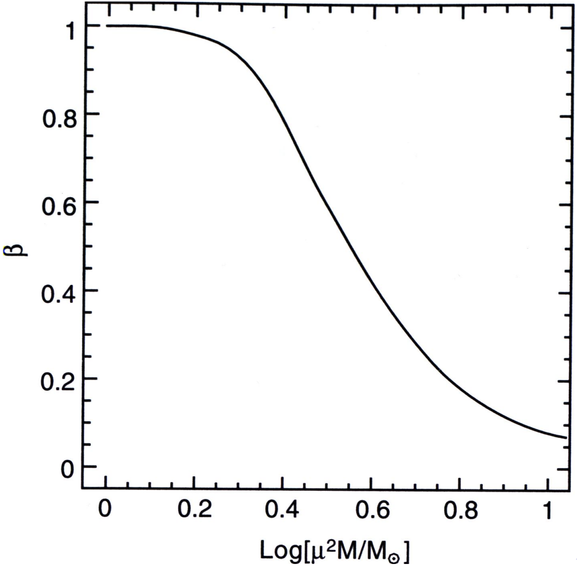
- $\Rightarrow$ polytrope of index $n = 3 \Rightarrow$

$$M = 4\pi M_3 \left(\frac{K}{\pi G}\right)^{3/2}$$

- $\Rightarrow$ *Eddington Quadratic Equation*

$$1 - \beta = 0.003 \left(\frac{M}{\mathrm{M}_\odot}\right)^2 \mu^4 \beta^4$$

Eddington Quadratic Equation



normalized solution for the Eddington quadratic equation

Properties Eddington Model

- for given composition (fixed μ), β decreases as M increases
- inserting the solution function $\beta(M,\mu)$ into $L = L^*_{\mathrm{edd}}(1-\beta)$ we obtain

$$L = 0.003\,\frac{4\pi c G \mathrm{M}_\odot}{\kappa_{\mathrm{surf}}}\mu^4\beta(M,\mu)^4\left(\frac{M}{\mathrm{M}_\odot}\right)^3$$

- recover mass-luminosity relation
- as star evolves, μ increases, hence it gets closer to Eddington limit and its luminosity rises;
  but unless the star is well mixed (e.g., fully convective), there will no longer be a uniform μ throughout the star

M-L Quiz Solution

For $M \to \infty$, recover the relation $L \propto M$ from

$$L = 0.003 \frac{4\pi c G \mathrm{M}_\odot}{\kappa_{\mathrm{surf}}} \mu^4 \beta(M,\mu)^4 \left(\frac{M}{\mathrm{M}_\odot}\right)^3$$

and

$$1 - \beta = 0.003 \left(\frac{M}{\mathrm{M}_\odot}\right)^2 \mu^4 \beta^4$$

In the second equation, the right hand side is finate $(0 < \beta < 1)$ hence as $M \to \infty$ on the right hand side $\beta \to 0$ is required. This menas that on the left hand side we can neglect β, and we have $\beta^4 \propto M^{-2}$. If we put this into the first equation, we are left with only one power in M, hence $L \propto M$.

Overview

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- local instabilities:
  - convection (Rayleigh Taylor instability)
  - semiconvection (layered convection)
  - thermohaline convection (salt finger instability)
  - rotation: e.g.:
    - shear instabilities (Kelvin-Helmholtz instability)
    - circulations (Eddington-Sweet)
    - etc.
  - magnetic instabilities: e.g.,
    - Parker instability,
    - dynamos,
    - etc.
- global instabilities
  - thermal instabilities
    - global
    - thin shell instability
  - dynamical instabilities

Virial Theorem: Ideal Gas without radiation

- Recall Virial Theorem:

$$3\int_0^M \frac{P}{\rho}\,\mathrm{d}m = -\Omega$$

- Ideal gas without radiation:

$$U = -\frac{1}{2}\Omega$$

- for the total energy of the star we hence have:

$$E = U + \Omega = \frac{1}{2}\Omega = -U$$

Quiz

What is the specific internal energy of gas and of radiation?

$$u_{\mathrm{gas}} = \frac{\mathcal{R}T}{\mu}\,, \qquad u_{\mathrm{rad}} = \frac{aT^4}{\rho}$$

Note:

$$P_{\mathrm{gas}} = \frac{\mathcal{R}T\rho}{\mu}\,, \qquad P_{\mathrm{rad}} = \frac{1}{3}aT^4$$

Virial Theorem: Ideal Gas *with* radiation

- Virial Theorem:

$$3\int_0^M \frac{P}{\rho}\,\mathrm{d}m = -\Omega$$

- Ideal gas *with* radiation:

$$\frac{P}{\rho} = \frac{P_{\mathrm{gas}}}{\rho} + \frac{P_{\mathrm{rad}}}{\rho} = \frac{\mathcal{R}}{\mu}T + \frac{a}{3\rho}T^4 = \frac{2}{3}u_{\mathrm{gas}} + \frac{1}{3}u_{\mathrm{rad}}$$

$$-\Omega = 3\int_0^M \frac{P}{\rho}\,\mathrm{d}m = 3\int_0^M \left(\frac{2}{3}u_{\mathrm{gas}} + \frac{1}{3}u_{\mathrm{rad}}\right)\mathrm{d}m = 2U_{\mathrm{gas}} + U_{\mathrm{rad}}$$

$$U_{\mathrm{gas}} = -\frac{1}{2}(\Omega + U_{\mathrm{rad}})$$

$\Rightarrow$ radiation reduces “effective” gravity

- the total energy hence is

$$E = \Omega + U_{\mathrm{rad}} + U_{\mathrm{gas}} = \frac{1}{2}(\Omega + U_{\mathrm{rad}}) = -U_{\mathrm{gas}}$$

Stellar Thermometer

- Rate of change of energy: $\dot{E} = L_{\rm nuc} - L$
- In thermal equilibrium: $L_{\rm nuc} = L$
- Recall: $\left(\frac{\partial \ln \varepsilon_{\rm nuc}}{\partial \ln T}\right)_\rho \geq 0$, $\left(\frac{\partial \ln \varepsilon_{\rm nuc}}{\partial \ln \rho}\right)_T \geq 0$
- What happens if there is an imbalance? (recall $E < 0$)
  - Case $L_{\rm nuc} > L$ ($\dot{E} > 0$):
    star expands, average $T \downarrow$, $\rho \downarrow$
    $\Rightarrow$ nuclear reaction rate decreases, $L_{\rm nuc} \downarrow$
  - Case $L_{\rm nuc} < L$ ($\dot{E} < 0$):
    star contracts, average $T \uparrow$, $\rho \uparrow$
    $\Rightarrow$ nuclear reaction rate increases, $L_{\rm nuc} \uparrow$
- $\Rightarrow$ self-regulation
- $\Rightarrow$ secular stability
- $\Rightarrow$ allows star to stay in thermal equilibrium for a long time.

Quiz

Can you think of a rection or nuclear process in a star where

- $$\left(\frac{\partial \ln \varepsilon_{\mathrm{nuc}}}{\partial \ln T}\right)_\rho = 0$$
- $$\left(\frac{\partial \ln \varepsilon_{\mathrm{nuc}}}{\partial \ln \rho}\right)_T = 0$$

Nuclear decays are (usually independent of T and ρ.

Thermal instability in degenerate gas

- In degenerate star (degenerate gas EOS)

$$P = K\rho^{5/3}$$

(for non-rel. deg. gas) pressure does not depend on T.
- $\Rightarrow$ no significant expansion, no drop in T or ρ (except due to the deviations from complete degeneracy)
- $T \uparrow \Rightarrow \varepsilon_{\text{nuc}} \uparrow \Rightarrow T \uparrow \Rightarrow \ldots$ thermonuclear runaway
- two possible outcomes are:
  - enough fuel is burned to unbind star (layers) $\Rightarrow$ supernova
  - degeneracy is “lifted” before star is unbound
    $\Rightarrow$ ideal gas EOS
    $\Rightarrow$ expansion
    $\Rightarrow$ cooling
    (e.g., nova, Type I X-ray Burst)

Thermal Stability Conditions

- In hydrostatic equilibrium

$$\frac{\mathrm{d}P_\mathrm{c}}{P_\mathrm{c}} = \frac{4}{3}\frac{\mathrm{d}\rho_\mathrm{c}}{\rho_\mathrm{c}}$$

- Using general EOS of form ($a = b = 1$ for ideal gas)

$$\frac{\mathrm{d}P}{P} = a\frac{\mathrm{d}\rho}{\rho} + b\frac{\mathrm{d}T}{T}$$

we obtain

$$\left(\frac{4}{3} - a\right)\frac{\mathrm{d}\rho_\mathrm{c}}{\rho_\mathrm{c}} = b\frac{\mathrm{d}T_\mathrm{c}}{T_\mathrm{c}}$$

- for $a < 4/3$ contraction causes heating (and vice versa)
- for degenerate gas $a \gtrsim 4/3$: expansion $\rightarrow$ heating $\Rightarrow$ unstable

Thin Shell Instability

- Assume thin shell of mass Δm, temperature T, density ρ, thickness l located inside star of radius R.
  Assume shell has a fixed lower boundary at r_0 and an upper boundary of r.
  Assume a the shell is "thin", i.e., $l = r - r_0 \ll r$
- In thermal equilibrium the energy that flows out of the shell is balanced by nuclear reactions

$$F(r) - F(r_0) = \int_{r_0}^{r} \varepsilon_{\mathrm{nuc}} 4\pi r^2 \rho \, \mathrm{d}r$$

- for a non-degenerate gas, if there is excess energy generation the shell will expand, or contract otherwise

Thin Shell Instability

- The mass of the shell is $\Delta m \approx 4\pi r_0^2 l \rho$, $(l = r - r_0)$ and therefore we have for the density

$$\frac{\mathrm{d}\rho}{\rho} = -\frac{\mathrm{d}l}{l} = -\frac{\mathrm{d}r}{l} = -\frac{\mathrm{d}r}{r}\frac{r}{l}$$

- in hydrostatic equilibrium, the pressure in the shell depends on the layers above and varies as r^{-4}:

$$\frac{\mathrm{d}P}{P} = -4\frac{\mathrm{d}r}{r} = 4\frac{l}{r}\frac{\mathrm{d}\rho}{\rho}$$

- using the general EOS we obtain

$$\left(4\frac{l}{r} - a\right)\frac{\mathrm{d}\rho}{\rho} = b\frac{\mathrm{d}T}{T}$$

- since $b > 0$, to have $\rho \downarrow \to T \downarrow$ we require $4l/r > a$
- For a thin shell $l/r \to 0$, hence $\rho \downarrow \to T \uparrow \Rightarrow$ instability!