Astrophysics I: Stars and Stellar Evolution

AST 4001

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Stars and Stellar Evolution, Fall 2008

Overview

1. Recap
   - Secular Thermal Stability (The Stellar Thermometer)
   - Degenerate Thermonuclear Runaway
   - Thin Shell Instability
2. Stellar Stability
   - Dynamical Stability
   - Cases of Dynamical (In)Stability

Overview

- local instabilities:
  - convection (Rayleigh Taylor instability)
  - semiconvection (layered convection)
  - thermohaline convection (salt finger instability)
  - rotation: e.g.:
    - shear instabilities (Kelvin-Helmholtz instability)
    - circulations (Eddington-Sweet)
    - etc.
  - magnetic instabilities: e.g.,
    - Parker instability,
    - dynamos,
    - etc.
- global instabilities
  - thermal instabilities
    - global
    - thin shell instability
  - dynamical instabilities

Stellar Thermometer

- Rate of change of energy: $\dot{E} = L_{\mathrm{nuc}} - L$
- In thermal equilibrium: $L_{\mathrm{nuc}} = L$
- Recall: $\left(\frac{\partial \ln \varepsilon_{\mathrm{nuc}}}{\partial \ln T}\right)_\rho \geq 0$, $\left(\frac{\partial \ln \varepsilon_{\mathrm{nuc}}}{\partial \ln \rho}\right)_T \geq 0$
- What happens if there is an imbalance? (recall $E < 0$)
  - Case $L_{\mathrm{nuc}} > L$ ($\dot{E} > 0$):
    star expands, average $T\downarrow$, $\rho\downarrow$
    $\Rightarrow$ nuclear reaction rate decreases, $L_{\mathrm{nuc}}\downarrow$
  - Case $L_{\mathrm{nuc}} < L$ ($\dot{E} < 0$):
    star contracts, average $T\uparrow$, $\rho\uparrow$
    $\Rightarrow$ nuclear reaction rate increases, $L_{\mathrm{nuc}}\uparrow$
- $\Rightarrow$ self-regulation
- $\Rightarrow$ secular stability
- $\Rightarrow$ allows star to stay in thermal equilibrium for a long time.

Thermal instability in degenerate gas

- In degenerate star (degenerate gas EOS)

$$P = K\rho^{5/3}$$

  (for non-rel. deg. gas) pressure does not depend on T.
- $\Rightarrow$ no significant expansion, no drop in T or ρ
  (except due to the deviations from complete degeneracy)
- $T \uparrow \Rightarrow \varepsilon_{\mathrm{nuc}} \uparrow \Rightarrow T \uparrow \Rightarrow \ldots$ thermonuclear runaway
- two possible outcomes are:
  - enough fuel is burned to unbind star (layers)
    $\Rightarrow$ supernova
  - degeneracy is "lifted" before star is unbound
    $\Rightarrow$ ideal gas EOS
    $\Rightarrow$ expansion
    $\Rightarrow$ cooling
    (e.g., nova, Type I X-ray Burst)

Thermal Stability Conditions

- In hydrostatic equilibrium

$$P_{\mathrm{c}} = \sqrt[3]{4\pi} B_n G M^{2/3} \rho_c^{4/3} \quad \Rightarrow \quad \frac{\mathrm{d}P_{\mathrm{c}}}{P_{\mathrm{c}}} = \frac{4}{3}\frac{\mathrm{d}\rho_{\mathrm{c}}}{\rho_{\mathrm{c}}}$$

- Using general EOS of form ($a = b = 1$ for ideal gas)

$$\frac{\mathrm{d}P}{P} = a\frac{\mathrm{d}\rho}{\rho} + b\frac{\mathrm{d}T}{T}$$

we obtain

$$\left(\frac{4}{3} - a\right)\frac{\mathrm{d}\rho_{\mathrm{c}}}{\rho_{\mathrm{c}}} = b\frac{\mathrm{d}T_{\mathrm{c}}}{T_{\mathrm{c}}}$$

- for $a < 4/3$ contraction causes heating (and vice versa)
- for degenerate gas $a \gtrsim 4/3$: expansion $\rightarrow$ heating $\Rightarrow$ unstable

Thin Shell Instability

- Assume thin shell of mass Δm, temperature T, density ρ, thickness l located inside star of radius R.
  Assume shell has a fixed lower boundary at r_0 and an upper boundary of r.
  Assume a the shell is "thin", i.e., $l = r - r_0 \ll r$
- In thermal equilibrium the energy that flows out of the shell is balanced by nuclear reactions

$$F(r) - F(r_0) = \int_{r_0}^{r} \varepsilon_{\mathrm{nuc}} 4\pi r^2 \rho \,\mathrm{d}r$$

- for a non-degenerate gas, if there is excess energy generation the shell will expand, or contract otherwise

Thin Shell Instability

- The mass of the shell is $\Delta m \approx 4\pi r_0^2 l \rho$, $(l = r - r_0)$ and therefore we have for the density

$$\frac{\mathrm{d}\rho}{\rho} = -\frac{\mathrm{d}l}{l} = -\frac{\mathrm{d}r}{l} = -\frac{\mathrm{d}r}{r}\frac{r}{l}$$

- in hydrostatic equilibrium, the pressure in the shell depends on the layers above and varies as r^{-4}:

$$\frac{\mathrm{d}P}{P} = -4\frac{\mathrm{d}r}{r} = 4\frac{l}{r}\frac{\mathrm{d}\rho}{\rho}$$

- using the general EOS we obtain

$$\left(4\frac{l}{r} - a\right)\frac{\mathrm{d}\rho}{\rho} = b\frac{\mathrm{d}T}{T}$$

- since $b > 0$, to have $\rho\downarrow\rightarrow T\downarrow$ we require $4l/r > a$
- For a thin shell $l/r \rightarrow 0$, hence $\rho\downarrow\rightarrow T\uparrow \Rightarrow$ instability!

Quiz

What happend if you heat a (non-rel.) degenerate thin shell?
What happens in a "plane parallel" approximation?

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Dynamical Stability

Assumptions and method:

- in hydrostatic equilibrium, pressure gradients balance gravity; this case we studied so far
- following our approach so far, we will look at radial perturbations (contraction and expansion of layers)
  $\Rightarrow$ assuming spherical symmetry
- Method of analysis:
  Will a temporary contraction (expansion) lead to restoration to original state or lead to further contraction (expansion)?

Dynamical Stability

- At point $m(r)$ the pressure is given by integrating the hydrostatic momentum equation, assuming $P(M) = 0$:

$$P_{\mathrm{h}}(m) = \int_m^M \frac{Gm}{4\pi r^4}\,\mathrm{d}m$$

- where, as usual, the density if given by

$$\rho = \frac{1}{4\pi r^2}\frac{\mathrm{d}m}{\mathrm{d}r}$$

- assume *relative* compression everywhere by ε, i.e., we have new radius coordinates r' defined by

$$r' = r - \varepsilon r = r(1-\varepsilon)$$

Dynamical Stability

Using for $\varepsilon \ll 1$ the approximation

$$(1 \pm \varepsilon)^x \approx 1 \pm x\varepsilon$$

from

$$\rho = \frac{1}{4\pi r^2}\frac{\mathrm{d}m}{\mathrm{d}r}$$

we obtain

$$\rho' = \frac{1}{4\pi(r(1-\varepsilon))^2}\frac{\mathrm{d}m}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}r'} = \frac{\rho}{(1-\varepsilon)^3} \approx \rho(1+3\varepsilon)$$

and for an adiabatic compression, $P \propto \rho^{\gamma_{\mathrm{ad}}}$,

$$P' = P\,(1+3\varepsilon)^{\gamma_{\mathrm{ad}}} \approx P\,(1+3\varepsilon\gamma_{\mathrm{ad}})$$

Dynamical Stability

For the hydrostatic equilibrium pressure,

$$P_{\mathrm{h}}(m) = \int_m^M \frac{Gm}{4\pi r^4}\,\mathrm{d}m\,,$$

we now obtain

$$P_{\mathrm{h}}{}'(m) = \int_m^M \frac{Gm}{4\pi(r(1-\varepsilon))^4}\,\mathrm{d}m \approx P_{\mathrm{h}}\,(1+4\varepsilon)$$

Dynamical Stability

- Initially, in the unperturbed hydrostatic star we have $P = P_\mathrm{h}$
- generally, we will find $P' \neq {P_\mathrm{h}}'$
- to restore equilibrium, for the case of compression, we need

$$P' > {P_\mathrm{h}}'$$

so that the star will re-expand, i.e.,

$$P\left(1 + 3\varepsilon\gamma_\mathrm{ad}\right) > P\left(1 + 4\varepsilon\right)$$

$$1 + 3\varepsilon\gamma_\mathrm{ad} > 1 + 4\varepsilon$$

$$\gamma_\mathrm{ad} > \frac{4}{3}$$

- this is the condition for dynamical stability
- the same result is obtained for expansion

Dynamical Stability

Notes:

- It can be shown that if $\gamma_{\text{ad}} > 4/3$ everywhere in the star, it is dynamically stable
- It is neutrally stable if $\gamma_{\text{ad}} = 4/3$ everywhere in the star
- global dynamical *instability* of the star results if

$$\int_0^M \left(\gamma_{\text{ad}} - \frac{4}{3}\right) \frac{P}{\rho} \, \mathrm{d}m < 0$$

- regions of high P/ρ (core) hence can dominate the global stability of the star

(In)Stability from the EOS

- simple equations of state:
  - ideal gas: $\gamma_{\mathrm{ad}} = 5/3 \Rightarrow$ stability
  - non-relativistic degenerate gas: $\gamma_{\mathrm{ad}} = 5/3 \Rightarrow$ stability
  - relativistic degenerate gas: $\gamma_{\mathrm{ad}} = 4/3 \Rightarrow$ neutral stability (Chandrasekhar limit)
  - pure radiation gas: $\gamma_{\mathrm{ad}} = 4/3 \Rightarrow$ neutral stability
- ideal gas with radiation

$$\gamma_{\mathrm{ad}} = \frac{5\beta^2 + 8(1-\beta)(4+\beta)}{3\beta^2 + 6(1-\beta)(4+\beta)}$$

  for $\beta \to 0$ we obtain $\gamma_{\mathrm{ad}} \to 4/3$ (radiation dominated)
- this is the case for star of increasingly higher mass

Stability of a Radiation Star

- Note that for pure radiation,

$$\frac{P}{\rho} = u_{\mathrm{rad}}/3\,,$$

and the viral theorem gives

$$-\Omega = 3\int_0^M \frac{P}{\rho}\,\mathrm{d}m = U_{\mathrm{rad}} = U$$

hence the total energy is

$$E = \Omega + U = 0$$

$\Rightarrow$ the star is neutrally bound
$\Rightarrow$ contraction or expansion does not cost energy!

Instability due to Ionization

- For the simple case of ionization of hydrogen gas we obtained the adiabatic index

$$\gamma_{\mathrm{ad}} = \frac{5 + \left(\frac{5}{2} + \frac{\chi}{k_{\mathrm{B}} T}\right)^2 x(1-x)}{3 + \left[\frac{3}{2} + \left(\frac{3}{2} + \frac{\chi}{k_{\mathrm{B}} T}\right)^2\right] x(1-x)}$$

  with a minimum at $x = 0.5$, and for the $\chi = 10 k_{\mathrm{B}} T$ we obtained $\gamma_{\mathrm{ad}} = 1.21$.

- for $x = 0$ and $x = 1$ we have $\gamma_{\mathrm{ad}} = 5/3$ (stable)
- for $\chi \approx k_{\mathrm{B}} T$ we find $\gamma_{\mathrm{ad}} < 4/3$ for $0.18 < x < 0.82$
- $\Rightarrow$ ionization can lead to instability
- however, interior of stars usually mostly ionized

Instability due other Processes

We recall

- electron-positron pair creation instability
- iron photo-disintegration
- helium photo-disintegration

Concluding, for very massive stars pressure increases stronger due to general relativity, i.e.,

$$P_{\mathrm{h},\mathrm{GR}}' > P_{\mathrm{h}}'$$

$\Rightarrow$ the critical value for γ_{ad} increases above $4/3$.
$\Rightarrow$ very massive radiation dominated star cannot be stable (mass limit: $\sim 100,000\,\mathrm{M}_\odot$)