Astrophysics I: Stars and Stellar Evolution

AST 4001

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Overview

Local Thermodynamic Equilibrium (Recap)

- Atmosphere is not in strict thermodynamic equilibrium (TE): temperature at bottom of a small volume element slightly different than at top
  $\Rightarrow$ gas temperature slightly different from radiation temperature
- We define local thermodynamic equilibrium (LTE) when T does not change much over mean free path of photon
- $\Rightarrow$ photon is absorbed at *almost* the same temperature as it was emitted
  $\Rightarrow$ gas temperature and radiation temperature are the same
- $\Rightarrow$ Kirchhoff's law applies
- However:
  radiation field is not isotropic
  net flux is not zero

Mean Free Path (Recap)

Assume a slab of matter with no emission and incident intensity $I_{\nu,0}$ and κ_ν independent of distance s from the surface of the slab.

The mean free path of a photon $\bar{s}$ is defined by

$$\bar{s} = \frac{\int_0^\infty s\, I_\nu \mathrm{d}s}{\int_0^\infty I_\nu \mathrm{d}s} = \frac{I_{\nu,0}\int_0^\infty s\, e^{-\kappa_\nu s}\mathrm{d}s}{I_{\nu,0}\int_0^\infty e^{-\kappa_\nu s}\mathrm{d}s} = -\frac{\mathrm{d}}{\mathrm{d}\kappa_\nu}\left(\ln \int_0^\infty e^{-\kappa s}\mathrm{d}s\right) = \frac{1}{\kappa_\nu}$$

That is, at location $\bar{s} = 1/\kappa_\nu$ the radiation has dropped to $1/e$ of the initial value.

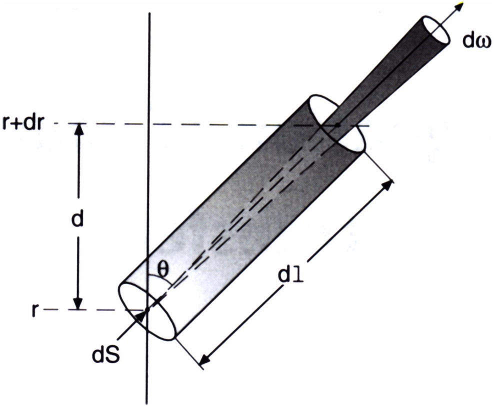
Quiz

At what distance has the radiation dropped by a factor 10?
By a factor 100?

Instructions:

- Work on this yourself and write down your solution (2 min)
- Discuss in groups of 2-3 (2 min)

Radiative Transfer



- cylindrical volume, angle Θ, bottom ara $\mathrm{d}S$, length $\mathrm{d}l$ at depth r $\Rightarrow \mathrm{d}l = \sec\Theta\,\mathrm{d}r$
- bean opening angle $\mathrm{d}\omega$
- frequency bin $\nu \ldots \nu + \mathrm{d}\nu$
- $\Rightarrow$ Energy going through the cylinder is $I_\nu\,\mathrm{d}\nu\,\mathrm{d}\omega$

Radiative Transfer

- for convenience we define distance from surface in opposite direction to r: $\mathrm{d}r = -\mathrm{d}z$
- the change in energy can then be written as $\mathrm{d}I_\nu \, \mathrm{d}\nu \, \mathrm{d}\omega$
- it has two contributions:
  - absorption:

$$-I_\nu \, \kappa_\nu \, \mathrm{d}l \, \mathrm{d}\nu \, \mathrm{d}\omega = +I_\nu \, \kappa_\nu \, \mathrm{d}z \sec\Theta \, \mathrm{d}\nu \, \mathrm{d}\omega$$

    where we used $\mathrm{d}l = -\sec\Theta \, \mathrm{d}z$
  - emission, using Kirchhoff's law:

$$j_\nu \, \mathrm{d}l \, \mathrm{d}\nu \, \mathrm{d}\omega = \kappa_\nu \, B_\nu(T) \, \mathrm{d}l \, \mathrm{d}\nu \, \mathrm{d}\omega = -\kappa_\nu \, B_\nu(T) \, \sec\Theta \, \mathrm{d}z \, \mathrm{d}\nu \, \mathrm{d}\omega$$

- The net change in intensity then is:

$$\mathrm{d}I_\nu(z,\Theta) = I_\nu(z,\Theta) \, \kappa_\nu \, \mathrm{d}z \, \sec\Theta - B_\nu(T) \, \kappa_\nu \, \mathrm{d}z \, \sec\Theta$$

Radiative Transfer

- Using the definition of optical depth inside the star τ_ν,

$$\mathrm{d}\tau_\nu = \kappa_\nu \, \mathrm{d}z$$

we can write

$$\mathrm{d}I_\nu(z,\Theta) = I_\nu(z,\Theta)\,\kappa_\nu\,\mathrm{d}z\,\sec\Theta - B_\nu(T)\,\kappa_\nu\,\mathrm{d}z\,\sec\Theta$$

in the form of the equation of transfer:

$$\cos\Theta\,\frac{\mathrm{d}I_\nu(z,\Theta)}{\mathrm{d}\tau_\nu} = I_\nu(z,\Theta) - B_\nu(T)$$

- Note: This simple LTE approximation assumes complete absorption of photon and re-emission in random direction; differential directional scattering is ignored. Good for many situations.

Radiative Equilibrium

- if the mass element under consideration has no net production or absorption of energy, in order to be in steady state, the total energy emitted in all directions from element ds *in* all frequencies

$$4\pi \int_0^\infty \kappa_\nu \, B_\nu(T) \, \mathrm{d}\nu \, \mathrm{d}s$$

has to equal the total energy absorbed *from* all directions by element ds:

$$\int_0^\infty \oint_{4\pi} \kappa_\nu \, I_\nu(z, \Theta) \, \mathrm{d}\omega \, \mathrm{d}\nu \, \mathrm{d}s$$

- Note: we neglect other forms of energy transport like conduction or convection

Radiation Moments

for further discussion we define three moments:

- mean intensity (0th moment):

$$J_\nu(z) = \frac{1}{4\pi} \oint_{4\pi} I_\nu(z,\Theta)\mathrm{d}\omega$$

- flux (1st moment):

$$F_\nu(z) = \oint_{4\pi} I_\nu(z,\Theta)\cos\Theta\mathrm{d}\omega$$

- 2nd moment:

$$K_\nu(z) = \frac{1}{4\pi} \oint_{4\pi} I_\nu(z,\Theta)\cos^2\Theta\mathrm{d}\omega$$

Radiative Equilibrium

- assuming κ_ν is independent of direction (isotropic) we can now write

$$\oint_{4\pi} \kappa_\nu I_\nu \mathrm{d}\omega\,\mathrm{d}s = \kappa_\nu \oint_{4\pi} I_\nu \mathrm{d}\omega\,\mathrm{d}s = 4\pi\,\kappa_\nu\,J_\nu\,\mathrm{d}s$$

- and the condition for radiative equilibrium becomes

$$4\pi \int_0^\infty \kappa_\nu\,B_\nu(T)\,\mathrm{d}\nu\,\mathrm{d}s = 4\pi \int_0^\infty \kappa_\nu\,J_\nu(z)\,\mathrm{d}\nu\,\mathrm{d}s$$

- or

$$\int_0^\infty \kappa_\nu\,[B_\nu(T) - J_\nu(z)]\,\mathrm{d}\nu = 0$$

Radiative Equilibrium

- introducing

$$\mu = \cos\Theta\,, \quad \mathrm{d}\mu = -\sin\Theta\,\mathrm{d}\Theta$$

- we can write the radiative flux

$$F_\nu = 2\pi \int_0^\pi I_\nu(z,\Theta)\,\sin\Theta\,\cos\Theta\,\mathrm{d}\Theta = 2\pi\int_{-1}^{+1} I_\nu(z,\mu)\,\mu\,\mathrm{d}\mu$$

- and the intensity

$$J_\nu = \frac{1}{4\pi}\oint_{4\pi} I_\nu(z,\Theta)\,\mathrm{d}\omega = \frac{1}{4\pi}\int_0^\pi 2\pi I_\nu(z,\Theta)\,\sin\Theta\,\mathrm{d}\Theta$$

$$J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(z,\mu)\,\mathrm{d}\mu$$

- and the equation of transfer simplifies to

$$\mu\,\frac{\mathrm{d}I_\nu(z,\mu)}{\kappa_\nu \mathrm{d}z} = I_\nu(z,\mu) - B_\nu(T)$$

Radiative Equilibrium

- integrating the transfer equation with respect to μ we obtain

$$\int_{-1}^{+1} \mu \, \frac{\mathrm{d}I_\nu(z,\mu)}{\kappa_\nu \mathrm{d}z} \, \mathrm{d}\mu = \frac{1}{\kappa_\mu} \frac{\mathrm{d}}{\mathrm{d}z} \int_{-1}^{+1} \mu \, I_\nu(z,\mu) \, \mathrm{d}\mu = \int_{-1}^{+1} [I_\nu(z,\mu) - B_\nu(T)] \, \mathrm{d}\mu$$

- substituting the flux we obtain

$$\frac{1}{2\pi\kappa_\nu} \frac{\mathrm{d}F_\nu(z)}{\mathrm{d}z} = \int_{-1}^{+1} I_\nu(z,\mu) \, \mathrm{d}\mu - \int_{-1}^{+1} B_\nu(T) \, \mathrm{d}\mu = 2\, J_\nu(z) - 2\, B_\nu(T)$$

- multiplying by $\kappa_\nu/2$ and integration over ν gives

$$\frac{1}{4\pi} \frac{\mathrm{d}}{\mathrm{d}z} \int_0^\infty F_\nu(z) \, \mathrm{d}\nu = \int_0^\infty \kappa_\nu [J_\nu(z) - B_\nu(T)] \, \mathrm{d}\nu = 0$$

- the last equality follows from radiative equilibrium.
- That is, the frequency-integrated flux $F(z) = \int_0^\infty F_\nu(z) \, \mathrm{d}\nu$ is independent of depth, $\mathrm{d}F/\mathrm{d}z = 0$.