Astrophysics I: Stars and Stellar Evolution

AST 4001

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Stars and Stellar Evolution, Fall 2008

Overview

1. Recap
   - Stellar Atmospheres
   - Radiative Transfer
   - Radiative Equilibrium

2. Stellar Atmosphere
   - Gray Atmosphere

Input for Atmosphere Model

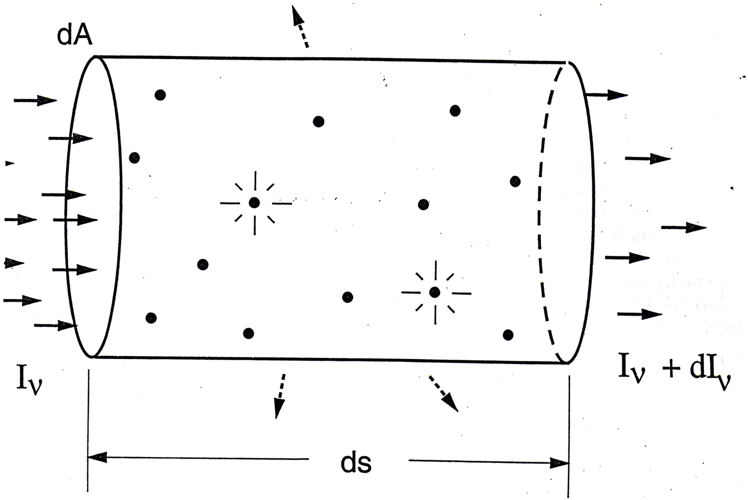
An atmosphere model determines T and ρ at the surface of the star as a function of depth.
As input parameters from the star we require

1. T_{eff}
2. $g = GM/R^2$
3. chemical composition (X, Y, Z), likely even the abundances of individual elements within Z

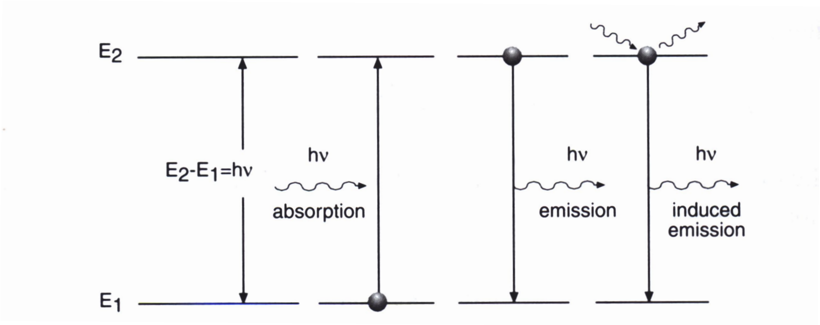
The output of an atmosphere model should provide the details of continuous and spectral energy distribution, colors, and angle dependence of the radiation field.

Generally, such a model is very complicated. In this class will examine some simplified models.

Scattering of photons



Absorption and Emission



Emission Coefficient

Given a frequency-dependent volume emission coefficient, j_ν, the energy that is emitted per unit volume $\mathrm{d}V$ per opening angle $\mathrm{d}\omega$ per frequency bin $\nu + \mathrm{d}\nu$ is given by

$$\mathrm{d}\epsilon_\nu = j_\nu \mathrm{d}\nu \mathrm{d}V \mathrm{d}\omega$$

If the emission is isotropic, the total energy emitted in all directions per second is then given by

$$4\pi \mathrm{d}V \int j_\nu \mathrm{d}\nu$$

Absorption Coefficient

Given an absorption coefficient κ_ν the initial intensity I_ν is reduced due to absorption by $\mathrm{d}I_\nu$ according to

$$\frac{\mathrm{d}I_\nu}{I_\nu} = -\kappa_\nu \mathrm{d}s = -\kappa_{\nu,\mathrm{M}}\rho \mathrm{d}s$$

where $\kappa_{\nu,\mathrm{M}}$ is called the mass absorption coefficient.
($[\kappa_{\nu,\mathrm{M}}] = \mathrm{cm}^2/\mathrm{g}$)
We define the optical depth τ at frequency ν by

$$\tau_\nu = \int \kappa_\nu \mathrm{d}s$$

or $\tau_\nu = \kappa_\nu s$ if κ_ν is independent of location. The intensity then drops as from its initial value $I_{\nu,0}$ according to extinction law

$$I_\nu = I_{\nu,0} e^{-\tau_\nu}$$

Kirchhoff's Law

- In strict *thermodynamic equilibrium* the total emission from a cylinder with base $\mathrm{d}A$ and thickness $\mathrm{d}s$, per $\mathrm{d}\omega$ and $\mathrm{d}\nu$ – $j_\nu\,\mathrm{d}\nu\,\mathrm{d}A\,\mathrm{d}s\,\mathrm{d}\omega$ – has to be equal to the absorption – $\mathrm{d}I_\nu\,\mathrm{d}A\,\mathrm{d}\omega\,\mathrm{d}\nu$.
- Using

$$\frac{\mathrm{d}I_\nu}{I_\nu} = -\kappa_\nu \mathrm{d}s$$

and the fact that in thermodynamic equilibrium the specific intensity $I_\nu = B_\nu$ (Planck function) we obtain

$$j_\nu = \kappa_\nu B_\nu(T)\ .$$

- This relation is called Kirchhoff's Law.

Local Thermodynamic Equilibrium (Recap)

- Atmosphere is not in strict thermodynamic equilibrium (TE): temperature at bottom of a small volume element slightly different than at top
  $\Rightarrow$ gas temperature slightly different from radiation temperature
- We define local thermodynamic equilibrium (LTE) when T does not change much over mean free path of photon
- $\Rightarrow$ photon is absorbed at *almost* the same temperature as it was emitted
  $\Rightarrow$ gas temperature and radiation temperature are the same
- $\Rightarrow$ Kirchhoff law applies
- However:
  radiation field is not isotropic
  net flux is not zero

Mean Free Path (Recap)

Assume a slab of matter with no emission and incident intensity $I_{\nu,0}$ and κ_ν independent of distance s from the surface of the slab.

The mean free path of a photon $\bar{s}$ is defined by

$$\bar{s} = \frac{\int_0^\infty s\, I_\nu \mathrm{d}s}{\int_0^\infty I_\nu \mathrm{d}s} = \frac{I_{\nu,0} \int_0^\infty s\, e^{-\kappa_\nu s} \mathrm{d}s}{I_{\nu,0} \int_0^\infty e^{-\kappa_\nu s} \mathrm{d}s} = -\frac{\mathrm{d}}{\mathrm{d}\kappa_\nu}\left(\ln \int_0^\infty e^{-\kappa s} \mathrm{d}s\right) = \frac{1}{\kappa_\nu}$$

That is, at location $\bar{s} = 1/\kappa_\nu$ the radiation has dropped to $1/e$ of the initial value.

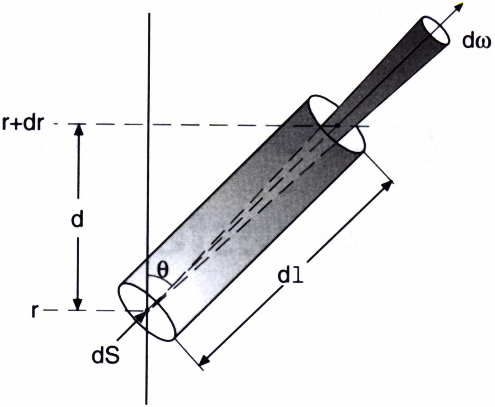
Quiz

At what distance has the radiation dropped by a factor 10?
By a factor 100?

Instructions:

- Work on this yourself and write down your solution (2 min)
- Discuss in groups of 2-3 (2 min)

Radiative Transfer



- cylindrical volume, angle Θ, bottom ara $\mathrm{d}S$, length $\mathrm{d}l$ at depth r
  $\Rightarrow \mathrm{d}l = \sec\Theta\,\mathrm{d}r$
- bean opening angle $\mathrm{d}\omega$
- frequency bin $\nu \ldots \nu + \mathrm{d}\nu$
- $\Rightarrow$ Energy going through the cylinder is $I_\nu\,\mathrm{d}\nu\,\mathrm{d}\omega$

Radiative Transfer

- for convenience we define distance from surface in opposite direction to r: $\mathrm{d}r = -\mathrm{d}z$
- the change in energy can then be written as $\mathrm{d}I_\nu\,\mathrm{d}\nu\,\mathrm{d}\omega$
- it has two contributions:
  - absorption:

$$-I_\nu\,\kappa_\nu\,\mathrm{d}l\,\mathrm{d}\nu\,\mathrm{d}\omega = +I_\nu\,\kappa_\nu\,\mathrm{d}z\sec\Theta\,\mathrm{d}\nu\,\mathrm{d}\omega$$

    where we used $\mathrm{d}l = -\sec\Theta\,\mathrm{d}z$
  - emission, using Kirchhoff's law:

$$j_\nu\,\mathrm{d}l\,\mathrm{d}\nu\,\mathrm{d}\omega = \kappa_\nu\,B_\nu(T)\,\mathrm{d}l\,\mathrm{d}\nu\,\mathrm{d}\omega = -\kappa_\nu\,B_\nu(T)\,\sec\Theta\,\mathrm{d}z\,\mathrm{d}\nu\,\mathrm{d}\omega$$

- The net change in intensity then is:

$$\mathrm{d}I_\nu(z,\Theta) = I_\nu(z,\Theta)\,\kappa_\nu\,\mathrm{d}z\,\sec\Theta - B_\nu(T)\,\kappa_\nu\,\mathrm{d}z\,\sec\Theta$$

Radiative Transfer

- Using the definition of optical depth inside the star τ_ν,

$$\mathrm{d}\tau_\nu = \kappa_\nu \, \mathrm{d}z$$

we can write

$$\mathrm{d}I_\nu(z,\Theta) = I_\nu(z,\Theta)\,\kappa_\nu\,\mathrm{d}z\,\sec\Theta - B_\nu(T)\,\kappa_\nu\,\mathrm{d}z\,\sec\Theta$$

in the form of the equation of transfer:

$$\cos\Theta\,\frac{\mathrm{d}I_\nu(z,\Theta)}{\mathrm{d}\tau_\nu} = I_\nu(z,\Theta) - B_\nu(T)$$

- Note: This simple LTE approximation assumes complete absorption of photon and re-emission in random direction; differential directional scattering is ignored. Good for many situations.

Radiative Equilibrium

- if the mass element under consideration has no net production or absorption of energy, in order to be in steady state, the total energy emitted in all directions from element ds *in* all frequencies

$$4\pi \int_0^\infty \kappa_\nu \, B_\nu(T) \, \mathrm{d}\nu \, \mathrm{d}s$$

has to equal the total energy absorbed *from* all directions by element ds:

$$\int_0^\infty \oint_{4\pi} \kappa_\nu \, I_\nu(z, \Theta) \, \mathrm{d}\omega \, \mathrm{d}\nu \, \mathrm{d}s$$

- Note: we neglect other forms of energy transport like conduction or convection

Radiation Moments

for further discussion we define three moments:

- mean intensity (0th moment):

$$J_\nu(z) = \frac{1}{4\pi} \oint_{4\pi} I_\nu(z,\Theta)\mathrm{d}\omega$$

- flux (1st moment):

$$F_\nu(z) = \oint_{4\pi} I_\nu(z,\Theta)\cos\Theta\mathrm{d}\omega$$

- 2nd moment:

$$K_\nu(z) = \frac{1}{4\pi} \oint_{4\pi} I_\nu(z,\Theta)\cos^2\Theta\mathrm{d}\omega$$

Radiative Equilibrium

- assuming κ_ν is independent of direction (isotropic) we can now write

$$\oint_{4\pi} \kappa_\nu I_\nu \mathrm{d}\omega\,\mathrm{d}s = \kappa_\nu \oint_{4\pi} I_\nu \mathrm{d}\omega\,\mathrm{d}s = 4\pi\,\kappa_\nu\,J_\nu\,\mathrm{d}s$$

- and the condition for radiative equilibrium becomes

$$4\pi \int_0^\infty \kappa_\nu\,B_\nu(T)\,\mathrm{d}\nu\,\mathrm{d}s = 4\pi \int_0^\infty \kappa_\nu\,J_\nu(z)\,\mathrm{d}\nu\,\mathrm{d}s$$

- or

$$\int_0^\infty \kappa_\nu\,[B_\nu(T) - J_\nu(z)]\,\mathrm{d}\nu = 0$$

Radiative Equilibrium

- introducing

$$\mu = \cos\Theta\,, \quad \mathrm{d}\mu = -\sin\Theta\,\mathrm{d}\Theta$$

- we can write the radiative flux

$$F_\nu = 2\pi \int_0^\pi I_\nu(z,\Theta)\,\sin\Theta\,\cos\Theta\,\mathrm{d}\Theta = 2\pi \int_{-1}^{+1} I_\nu(z,\mu)\,\mu\,\mathrm{d}\mu$$

- and the intensity

$$J_\nu = \frac{1}{4\pi} \oint_{4\pi} I_\nu(z,\Theta)\,\mathrm{d}\omega = \frac{1}{4\pi} \int_0^\pi 2\pi I_\nu(z,\Theta)\,\sin\Theta\,\mathrm{d}\Theta$$

$$J_\nu = \frac{1}{2} \int_{-1}^{+1} I_\nu(z,\mu)\,\mathrm{d}\mu$$

- and the equation of transfer simplifies to

$$\mu\,\frac{\mathrm{d}I_\nu(z,\mu)}{\kappa_\nu \mathrm{d}z} = I_\nu(z,\mu) - B_\nu(T)$$

Radiative Equilibrium

- integrating the transfer equation with respect to μ we obtain

$$\int_{-1}^{+1} \mu \frac{\mathrm{d}I_\nu(z,\mu)}{\kappa_\nu \mathrm{d}z}\,\mathrm{d}\mu = \frac{1}{\kappa_\mu}\frac{\mathrm{d}}{\mathrm{d}z}\int_{-1}^{+1} \mu\, I_\nu(z,\mu)\,\mathrm{d}\mu = \int_{-1}^{+1} [I_\nu(z,\mu) - B_\nu(T)]\,\mathrm{d}\mu$$

- substituting the flux we obtain

$$\frac{1}{2\pi\kappa_\nu}\frac{\mathrm{d}F_\nu(z)}{\mathrm{d}z} = \int_{-1}^{+1} I_\nu(z,\mu)\,\mathrm{d}\mu - \int_{-1}^{+1} B_\nu(T)\,\mathrm{d}\mu = 2\,J_\nu(z) - 2\,B_\nu(T)$$

- multiplying by $\kappa_\nu/2$ and integration over ν gives

$$\frac{1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}z}\int_0^\infty F_\nu(z)\,\mathrm{d}\nu = \int_0^\infty \kappa_\nu[J_\nu(z) - B_\nu(T)]\,\mathrm{d}\nu = 0$$

- the last equality follows from radiative equilibrium.
- That is, the frequency-integrated flux $F(z) = \int_0^\infty F_\nu(z)\,\mathrm{d}\nu$ is independent of depth, $\mathrm{d}F/\mathrm{d}z = 0$.

Summary of Temperature in Atmosphere

- atmosphere in local thermodynamic equilibrium
  Kirchhoff's law applies: $j_\nu = \kappa_\nu B_\nu(T)$
- plane parallel atmosphere with a thickness much smaller than radius of star
  curvature can be neglected
- assume gray atmosphere, i.e., a suitable average absorption coefficient $\bar{\kappa}$ can be found so that all quantities can be integrated
- the atmosphere is in radiative equilibrium
  no net energy is generated or absorbed (consumed).

Overview

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Radiative Transfer in Atmosphere

- compute the first moment of the transfer equation

$$\cos\Theta\,\frac{\mathrm{d}I_\nu(z,\Theta)}{\mathrm{d}\tau_\nu} = I_\nu(z,\Theta) - B_\nu(T)$$

  by multiplication with $\cos\Theta$ and integration over all solid angles

$$\oint_{4\pi}\cos^2\Theta\,\frac{\mathrm{d}I_\nu(z,\Theta)}{\mathrm{d}\tau_\nu}\,\mathrm{d}\omega = \frac{\mathrm{d}}{\mathrm{d}\tau_\nu}\oint_{4\pi}\cos^2\Theta\,I_\nu(z,\Theta)\,\mathrm{d}\omega = \ldots$$

$$\ldots = \oint_{4\pi}\cos\Theta\,I_\nu(z,\Theta)\,\mathrm{d}\omega - \oint_{4\pi}\cos\Theta\,B_\nu(T)\,\mathrm{d}\omega$$

- Note that because $B_\nu(T)$ is isotropic, the last term vanishes and we obtain

$$\frac{\mathrm{d}K_\nu(z)}{\mathrm{d}\tau_\nu} = \frac{F_\nu(z)}{4\pi}$$

Radiative Transfer in Gray Atmosphere

- define mean opacity $\bar{\kappa}$ such that we obtain a mean optical depth τ by

$$\mathrm{d}\tau = \bar{\kappa}\,\mathrm{d}z$$

- The frequency integral of the first moment of the transfer equation,

$$\int_0^\infty \frac{\mathrm{d}K_\nu(z)}{\mathrm{d}\tau_\nu}\,\mathrm{d}\nu = \int_0^\infty \frac{F_\nu(z)}{4\pi}\,\mathrm{d}\nu$$

then becomes

$$\frac{\mathrm{d}K(z)}{\mathrm{d}\tau} = \frac{F(z)}{4\pi} = \frac{F}{4\pi}$$

Radiative Transfer in Gray Atmosphere

- differentiation with regard to τ yields in radiative equilibrium

$$\frac{\mathrm{d}^2 K(z)}{\mathrm{d}\tau^2} = \frac{1}{4\pi}\frac{\mathrm{d}F}{\mathrm{d}\tau} = J - B = 0$$

- where J and B are now frequency-integrated quantities.
- to evaluate K we will assume that I is isotropic; since we only multiply it with a positive quantity, $\cos^2\Theta$, there will be no effect from almost, but not quite, cancellation of two large quantities (at top and bottom) as it is in the case of the flux
- we hence can approximate from the definition of K

$$K = \frac{1}{4\pi}\oint_{4\pi} I\,\cos^2\Theta\,\mathrm{d}\omega = \frac{1}{2}J\int_0^{\pi}\cos^2\Theta\,\sin\Theta\,\mathrm{d}\Theta = \frac{1}{3}J$$

(Eddington approximation) [yet another]

Radiative Transfer in Gray Atmosphere

- we can now use $K = \frac{1}{3}J$ in the first moment of the transfer equation,

$$\frac{\mathrm{d}K(z)}{\mathrm{d}\tau} = \frac{F}{4\pi}$$

and obtain

$$\frac{\mathrm{d}J(z)}{\mathrm{d}\tau} = \frac{3}{4\pi}F$$

- integration with regards to τ then gives

$$J = \frac{3}{4\pi}F\,\tau + \mathrm{const.}$$

- But we also have from the definition of T_{eff}, which is considered to be a constant:

$$J = B = \frac{\sigma}{\pi}T^4 = \frac{3}{4}\frac{\sigma}{\pi}T_{\mathrm{eff}}^4(\tau + c_3)$$

Radiative Transfer in Gray Atmosphere

- to derive the constant c_3 we consider that at the surface, $\tau = 0$, there is no inward flux, but we assume that in our approximation the intensity at the surface is independent of direction. Let us call the intensity at the surface I_0^+.
- at the surface we then have

$$J(0) = \frac{2\pi}{4\pi} \int_0^{\pi/2} I(0) \sin\Theta \, d\Theta = \frac{I_0^+}{2}$$

$$F(0) = 2\pi \int_0^{\pi/2} I(0) \cos\Theta \sin\Theta \, d\Theta = \pi I_0^+$$

and hence

$$J(0) = \frac{F(0)}{2\pi} = B(0) = \frac{\sigma T^4(0)}{\pi} = \frac{1}{2\pi} \sigma \, T_{\mathrm{eff}}^4$$

Radiative Transfer in Gray Atmosphere

- therefore, at the surface we now have

$$T^4(0) = \frac{1}{2} T_{\text{eff}}^4$$

- from

$$\frac{\sigma}{\pi} T^4 = \frac{3}{4} \frac{\sigma}{\pi} T_{\text{eff}}^4 (\tau + c_3)$$

we obtain

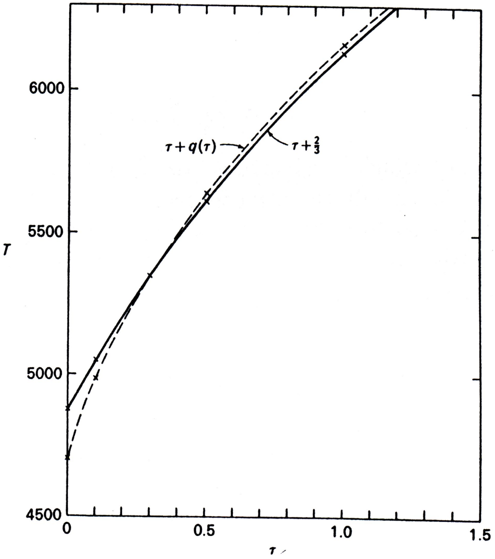
$$\frac{1}{2} T_{\text{eff}}^4 = \frac{3}{4} T_{\text{eff}}^4 \, c_3 \quad \Rightarrow \quad c_3 = \frac{2}{3}$$

- and the final distribution of temperature in a gray atmosphere is

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \left(\tau + \frac{2}{3} \right)$$

- Note that $T = T_{\text{eff}}$ at $\tau = 2/3$

Approximation Compared with True Stratification



more generally we can write

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \left(\tau + q(\tau)\right)$$

Quiz

Compute the moments of I: J, F, and K

- in the center of the star
- in thermodynamic equilibrium (TE)

Instructions:

- Work on this yourself and write down your solution (2 min)
- Discuss in groups of 2-3 (2 min)