RECAP

LANE EMDEN EQN:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Theta}{d\xi}\right) = -\Theta^n$$

$$\rho = \rho_c \cdot \Theta^n$$

$$r = \alpha \cdot \xi \quad , \quad \alpha = \sqrt{\frac{(n+1)K}{4\pi G \rho_c^{\left(\frac{n-1}{n}\right)}}}$$

$$\boxed{P = K\rho^{1+\frac{1}{n}}}$$

HW: SET UP MESA

RUN 15 $M_\odot$ STAR, SOLAR

25 ?

INTEGRATE LANE-EMDEN EQN.

ASSUME $\rho_c = 0.1$

IDEAL GAS + [RAD?]

COMPOSITION: 50% H, 50% He by mass

CAN WE DETERMINE L (I HAVE NEVER TRIED)

ASSUME $\tau = 2/3$

WRITE FORTRAN PROGRAM

DETERMINE $\Theta''|_{\xi=0}$ needed?

SOLUTIONS TO LANE EMDEN EQUN
(POLYTROPIC STELLAR MODELS)

RADIUS OF STAR

$$\boxed{R = \alpha \cdot \xi_1}$$

$\Theta(\xi_1) \overset{!}{=} 0$ DEFINES ξ_1 ← "ONE"

MASS OF STAR:

$$M = \int_0^R 4\pi r^2 \rho \, dr = 4\pi\alpha^3 \int_0^{\xi_1} \xi^2 \Theta^n \, d\xi \quad \Bigg| \begin{array}{l} r = \alpha \cdot \xi \\ \rho = \rho_c \Theta^n \end{array}$$

USE LE: $\quad \xi^2 \Theta^n = -\frac{d}{d\xi}\left(\xi^2 \frac{d\Theta}{d\xi}\right)$

$$M = -4\pi\alpha^3 \rho_c \int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2 \frac{d\Theta}{d\xi}\right) d\xi$$

$$= -4\pi\alpha^3 \rho_c \xi_1^2 \left(\frac{d\Theta}{d\xi}\right)_{\xi_1}$$

DEF. $\quad M_n := -\xi_1^2 \left(\frac{d\Theta}{d\xi}\right)\Big|_{\xi_1} > 0 \quad$ WITH ξ_1 AND $\frac{d\Theta}{d\xi}\Big|_{\xi_1}$ FROM SLN OF LE

$$R_n := \xi_1$$

$$\rightarrow \boxed{\begin{array}{l} M = 4\pi\alpha^3 \rho_c M_n \\ R = \alpha \cdot R_n \end{array}}$$

POLYTROPIC MASS-RADIUS RELATION.

$$M = 4\pi\alpha^3 \rho_c M_n \longrightarrow \text{eliminate } \rho_c$$

using $\alpha^2 = \dfrac{(n+1)K}{4\pi G \rho_c^{\frac{n-1}{n}}}$

$$M = 4\pi\alpha^3 \left(\frac{(n+1)K}{4\pi G \alpha^2}\right)^{\frac{n}{n-1}} M_n$$

eliminate α using $R = \alpha \cdot R_n \longrightarrow \alpha = R/R_n$

$$\left(\frac{GM}{M_n}\right)^{n-1} = \frac{(4\pi)^{n-1-n}\,\alpha^{3n-3-2n}}{G^{n-(n-1)}}\left[(n+1)K\right]^n \frac{\alpha^{n-3}}{4\pi G}\left[(n+1)K\right]$$

$$\left(\frac{GM}{M_n}\right)^{n-1} = \left(\frac{R}{R_n}\right)^{n-3} \left[\frac{(n+1)K}{4\pi G}\right]^n \quad \text{OR}$$

$$\boxed{\left(\frac{GM}{M_n}\right)^{n-1} = \left(\frac{R}{R_n}\right)^{3-n} \left[\frac{(n+1)K}{4\pi G}\right]^n}$$

POLYTROPIC MASS-RADIUS RELATION

FOR $\boxed{n = 3}$: MASS INDEP OF R
→ ONLY DEP ON K

$$M = 4\pi M_3 \left(\frac{K}{\pi G}\right)^{3/2}$$

→ ONLY ONE POSSIBLE MASS SATISFIES Equilibrium

FOR $\boxed{n=1}$ R becomes indep of mass and determined by K:

$$R = 4\pi R_1 \left(\frac{K}{2\pi G}\right)^{1/2}$$

FOR $1 < n < 3$: $R^{3-n} \sim M^{1-n}$

Polytropic Constants

(we already have R_n & M_n)

Density:

$$M = -4\pi\alpha^3 \rho_c \xi_1^2 \left.\frac{d\theta}{d\xi}\right|_{\xi_1}$$

$$\rho_c = -\frac{M}{4\pi\alpha^3 \xi_1^2 \left.\frac{d\theta}{d\xi}\right|_{\xi_1}} = \underbrace{\frac{3M}{4\pi R^3}}_{\bar{\rho}} \underbrace{\frac{1}{\frac{3}{\xi_1}\left(\left.\frac{d\theta}{d\xi}\right|_{\xi_1}\right)}}_{=:D_n}$$

$\rightarrow \rho_c = \text{const.}\ \bar{\rho}$ for given $\underline{n}$!

Pressure:

$$P = K \cdot \rho^{\frac{n+1}{n}}$$

$$\rightarrow P_c = \frac{(4\pi G)^{1/n}}{n+1}\left(\frac{GM}{M_n}\right)^{\frac{n-1}{n}}\left(\frac{R}{R_n}\right)^{\frac{3-n}{n}} \rho_c^{\frac{n+1}{n}}$$

$$\boxed{P_c = \sqrt[3]{4\pi}\, \underset{\substack{\uparrow \\ \sim 0.1 \dots 0.2 \text{ (varies with } n)}}{B_n}\, G M^{2/3} \rho_c^{4/3}}$$

Almost universal for $\underline{\text{polytropic}}$ stars

WD MASS - RADIUS RELATION

$M \sim M_\odot$

$R \sim R_{EARTH}$

$X = C, O, Mg \rightarrow \mu_e \simeq 2$

$\rightarrow$ NON-REL DEG EOS

$$P_{e,deg} = K_1 \rho^{5/3}$$

$$\downarrow \quad \downarrow$$

$$K \quad n = 1.5$$

M-R REL:

$$\left(\frac{GM}{M_n}\right)^{n-1} \left(\frac{R}{R_N}\right)^{3-n} = \frac{\left[(n+1)K\right]^n}{4\pi G}$$

$\rightarrow R \sim M^{-1/3}$, $\rho \sim M R^{-3} \sim M^2$

$\rightarrow$ FOR $M \uparrow \rightarrow R \downarrow$

what happens if we increase M?

$\rightarrow$ no longer non-rel deg EOS!

DEGENERATE e^- EOS:

$$P_{e,\,rel,\,deg} = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3} \frac{1}{u^{4/3}} \left(\frac{\rho}{\mu_e}\right)^{4/3}$$

ATOMIC MASS UNIT

$$=: K_2 \cdot \rho^{4/3}$$

$$K_2 \to K \qquad \rho^{4/3} \to n=3$$

BUT: THERE IS A UNIQUE MASS FOR $n=3$

$$M = 4\pi\, M_3 \left(\frac{K}{\pi G}\right)^{3/2}$$

$\longrightarrow$ MAX MASS OF WD!

RECALL

$$\mu_e = \left(\sum_i X_i \frac{Z_i}{A_i}\right)^{-1} = \left(\sum Y_i Z_i\right)^{-1}$$

$$Y_i = X_i / A_i$$

$$\mu_e \simeq \left(X + \tfrac{1}{2}Y + \tfrac{1}{2}(1-X-Y)\right)^{-1} = \frac{2}{1+X}$$

$\uparrow$
$\sim$

THIS MASS IS CALLED

CHANDRASEKHAR MASS

$$M_{CH} = \frac{M_3}{4\pi}\left(\frac{3}{2}\right)^{1/2}\left(\frac{hc}{G u^{4/3}}\right)^{3/2} \mu_e^{-2}$$

$$= 5.836\, M_\odot \cdot \mu_e^{-2} \quad , \mu_e \simeq 2$$

$$M_{CH} = 1.459\, M_\odot \cdot \left(\frac{\mu_e}{2}\right)^{-2}$$

NOBEL PRIZE 1983

FOR IRON CORE : $\mu_e \sim 2.15$

$$\longrightarrow M_{CH} \sim 1.26\, M_\odot$$

- "HOT" CORE ONLY PARTIALLY DEGENERATE
  - $\rightarrow$ PARTIAL DEG. REL EOS
  - $\rightarrow$ $M_{CRIT} > M_{CH}$

$$M_{CRIT} = M_{CH} \cdot \left[1 + \frac{\pi^2 k_B^2 T^2}{\varepsilon_F^2}\right]$$

$$\varepsilon_F = 1.11\left(\frac{\rho}{10^7\, g\, cm^{-3}} Y_e\right)^{1/3} MeV \quad , Y_e = \frac{1}{\mu_e}$$