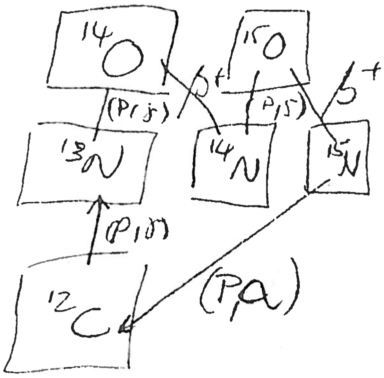
EXERCISE

HOT CNO CYCLE LIMIT

(X-RAY BURST / NOVA)



$\tau_{1/2,\,^{14}O} = 70\,s$

$\tau_{1/2,\,^{15}O} = 123\,s$

$\boxed{T_7 \gtrsim 8}$

ASSUME

$\tau_{p,\gamma},\ \tau_{p,\alpha} \ll \tau_{\beta^+}$

INITIAL ABUNDANCE

12% ^{12}C

28.8% ^{4}He

70% ^{1}H

[→ WRITE DOWN $\frac{\partial}{\partial t} Y_i$ EQN]

a) WHAT IS EQUILIBRIUM RATIO OF

$Y_{^{14}O} / Y_{^{15}O}$?

b) HOW LONG DOES IT TAKE TO BURN ALL ^{1}H TO ^{4}He ?

c) What is the mass fraction of stable (long) isotopes of the cycle has completed?

SOLUTION

$$Y_{^{14}O} = Y_{^{12}C}^{INIT} \cdot \frac{\tau_{1/2}(^{14}O)}{\tau_{1/2}(^{15}O) + \tau_{1/2}(^{14}O)}$$

$$Y_{^{1}H}^{INIT} = \frac{70\%}{1} = 0.7$$

$$T = \frac{Y_{^{1}H}^{INIT}}{Y_{^{12}C}^{INIT}} \frac{1}{4 \cdot \ln 2} \frac{\tau_{1/2}(^{14}O)}{\tau_{1/2}(^{14}O)} \left(\tau_{1/2}(^{14}O) + \tau_{1/2}(^{15}O)\right).$$

$$= \frac{0.7}{0.001} \cdot \frac{1}{2.77} \cdot 1 \cdot 193\,s$$

$$= 700 \cdot 0.36 \cdot 193\,s = 4.87 \times 10^{4}\,s$$

$$= 13.5\,h$$

NOTE THE FACTOR $\frac{1}{\ln 2}$

C) $^{14}O(\beta^+)\,^{14}N$ STABLE

$^{15}O(\beta^+)\,^{15}N$ STABLE

$$\rightarrow X(^{15}N) = 15 \cdot \frac{\tau_{1/2}(^{15}O)}{\tau_{1/2}(^{15}O) + \tau_{1/2}(^{14}O)} \cdot Y_{^{12}C}^{INI} = \frac{123}{193} \cdot 0.001 \cdot 15 = 9.56 \times 10^{-3}$$

$$X(^{14}N) = 14 \cdot \frac{\tau_{1/2}(^{14}O)}{\tau_{1/2}(^{15}O) + \tau_{1/2}(^{14}O)} \cdot Y_{^{12}C}^{INI} = \frac{70}{193} \cdot 0.001 \cdot 14 = 5.08 \times 10^{-3}$$

$$X(^{4}H) = 1 - X(^{15}N) - X(^{14}N) = 9.8536 \times 10^{-1}$$

SOLUTION

a)
$$\frac{Y(^{14}O)}{Y(^{15}O)} = \frac{\tau_{^{14}O}}{\tau_{^{15}O}} = \frac{70s}{123s} = \ldots$$

b)
$$\lambda_{^{14}O(\beta^+)} = \frac{\ln 2}{\tau_{1/2}(^{14}O)}$$

also given this

$$\frac{\partial}{\partial t} Y_{^1H} = -4 \cdot \lambda_{^{14}O(\beta^+)} \cdot Y_{^{14}O}$$

Separation of variables

$$= -4\, \lambda_{^{15}O(\beta^+)}\, Y_{^{15}O}$$

BOTH GIVE THE SAME!

4 ^1H USED IN EACH CYCLE

Separation of VARIABLES

$$Y_{^1H}(t=0) = T \cdot 4 \cdot \frac{\ln 2}{\tau_{1/2}(^{14}O)} \cdot Y_{^{14}O}$$

$$Y_{^{14}O} + Y_{^{15}O} = Y_{^{12}C}^{\text{INITIAL}} = .0001 \left(= \frac{1.2\%}{12}\right)$$

$$\frac{Y_{^{14}O}}{Y_{^{14}O} + Y_{^{15}O}} = \frac{70s}{123s + 70s} = \frac{70}{193} \simeq 0.3627 = \frac{\tau_{1/2}(^{14}O)}{\tau_{1/2}(^{14}O) + \tau_{1/2}(^{15}O)}$$

JOTE :

THE FINAL $X(^4He)$ IS NOT JUST THE SUM OF INITIAL $X(^1H) + X(^4He)$

THE REASON IS THAT SOME OF THE NUCLEONS (PROTONS) ARE NOW PART OF THE RESIDUAL ^{14}N AND ^{15}N IN EXCESS OF # NUCLEON THAT WERE IN ^{12}C

CONSIDER $\quad {}^{12}C(p,\gamma)\,{}^{13}N$

$$\frac{dY_{^{12}C}}{dt} = -\lambda_{^{12}C(p,\gamma)^{13}N}\, Y_{^{12}C}\, Y_{^{1}H}$$

$$\frac{dY_{^{1}H}}{dt} = -\lambda_{^{12}C\ldots}\, Y_{^{12}C}\, Y_{^{1}H}$$

$$\frac{dY_{^{13}N}}{dt} = +\lambda_{\ldots}\, Y_{^{12}C}\, Y_{^{1}H}$$

(1) TIME DISCRETIZATION

$$\Delta Y_{^{12}C} = -\lambda_{^{12}C\ldots}\, Y_{^{12}C}\, Y_{^{1}H}\, \Delta t$$

$$\Delta Y_{^{1}H} = -\ldots$$

$$\Delta Y_{^{13}N} = +\ldots$$

Q: What happens if Δt too large?

WITH $\Delta Y = Y^{t+\Delta t} - Y^{t}$

NOW: IMPLICIT SOLUTION

USE ABONDANCES @ END OF TIME STEP !

$$Y^{+} = Y^{0} + \Delta Y \qquad \text{(IMPLICIT)}$$

$$\Delta Y_{^{12}C} = -\lambda_{^{12}C}\Delta t\left(Y^{0}_{^{12}C} + \Delta Y_{^{12}C}\right)\left(Y^{0}_{^{1}H} + \Delta Y_{^{1}H}\right)$$

$$= -\lambda_{^{12}C}\Delta t\Big(Y^{0}_{^{12}C}Y^{0}_{^{1}H} + Y^{0}_{^{12}C}\Delta Y_{^{1}H} + Y_{^{1}H}\Delta Y_{^{12}C}$$

$$+ \underbrace{\Delta Y_{^{12}C}\Delta Y_{^{1}H}}_{\to\ \text{neglect (QUADRATIC)}}\Big)\Delta t$$

$\longrightarrow$ LINEARIZE

SIMILAR $\Delta Y_{^{1}H}$, $\Delta Y_{^{13}N}$

$$\Delta Y_{^{12}C}\left(1 + \lambda_{^{12}C}\Delta t\, Y_{^{1}H}\right) + \Delta Y_{^{1}H}\left(\lambda_{^{12}C}\Delta t\, Y^{0}_{^{12}C}\right)$$

$$= -\lambda_{^{12}C}Y_{^{12}C}Y_{^{1}H}\Delta t$$

AND

$$\Delta Y_{^{1}H}\left(1 + \lambda_{^{12}C}\Delta t\, Y_{^{12}C}\right) + \Delta Y_{^{12}C}\left(\lambda_{^{12}C}\Delta t\, Y^{0}_{^{1}H}\right) = \ldots$$

$$-\Delta Y_{^{13}N} + \lambda_{^{12}C}\Delta t\,\Delta Y_{^{12}C}Y^{0}_{^{1}H} + \lambda_{^{12}C}\Delta t\, Y^{0}_{^{12}C}\Delta Y_{^{1}H} = \ldots \qquad |\times(-1)$$