Vacuum Oscillation

$$|\nu\rangle_0 = \sum_\alpha a_\alpha(0)\,|\nu_\alpha\rangle_0 = \sum_k a_k(0)\,|\nu_k\rangle_0 \quad . \quad \begin{cases} |\nu_\alpha\rangle_0 = \sum_k U_{\alpha k}\,|\nu_k\rangle_0 \\ a_k(0) = \sum_\alpha U_{\alpha k}\,a_\alpha(0) \end{cases}$$

$$\rightarrow \quad |\nu_i\rangle_{\Delta t} = (1 - iE_i\Delta t)\,|\nu_i\rangle_0$$

$$\rightarrow \quad |\nu\rangle_{\Delta t} = \sum_k a_k(0)\,|\nu_k\rangle_{\Delta t} = \sum_k a_k(0)\,(1 - iE_k\Delta t)\,|\nu_k\rangle_0$$

$$= \sum_{k\alpha} a_k(0)\,(1 - iE_k\Delta t)\,U^*_{\alpha k}\,|\nu_\alpha\rangle_0$$

$$= \sum_{k\alpha\beta} a_\beta(0)\,(1 - iE_k\Delta t)\,U_{\beta k}\,U^*_{\alpha k}\,|\nu_\alpha\rangle_0$$

$$\rightarrow \quad a_\alpha(\Delta t) = \sum_{k\beta} a_\beta(0)\,(1 - iE_k\Delta t)\,U_{\beta k}U^*_{\alpha k}$$

$$= a_\alpha(0) - \sum_{k\beta} iE_k\Delta t\, U_{\beta k}U^*_{\alpha k}\,a_\beta(0)$$

$$\rightarrow \quad i\frac{d\,a_\alpha(t)}{dt} = \sum_{k\beta} E_k U_{\beta k}U^*_{\alpha k}\,a_\beta(t) \quad , \quad E_k \simeq p + \frac{m_k^2}{2p}$$

$$= \sum_{k\beta}\left(p + \frac{m_k^2}{2p}\right) U_{\beta k}U^*_{\alpha k}\,a_\beta(t)$$

$$= p\,a_\alpha(t) + \sum_{k\beta}\frac{m_k^2}{2p}\,U_{\beta k}U^*_{\alpha k}\,a_\beta(t)$$

$$\Psi \equiv \begin{pmatrix} a_e^* \\ a_\mu^* \\ a_\tau^* \\ \vdots \end{pmatrix}$$

$$\rightarrow \quad i\frac{d}{dt}\Psi = \cancel{p\Psi} + \underbrace{UMU^+}_{\equiv\, H_v}\Psi \quad , \quad M = \frac{1}{2p}\begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & \ddots \end{pmatrix}$$

for observable.

example :

- 2 generation :

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad |\nu\rangle_0 = |\nu_e\rangle \quad \Rightarrow \quad a_e = a_e^* = 1, \quad a_x = 0$$

$$H_\nu = \cancel{\frac{m_1^2 + m_2^2}{4p}\mathbb{I}} + \frac{-\Delta m^2}{4p}\begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \quad \Delta m^2 \equiv m_2^2 - m_1^2$$

$$= \frac{-\Delta m^2}{4p}\begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} c & -s \\ -s & -c \end{pmatrix} = \frac{-\Delta m^2}{4p}\begin{pmatrix} c^2 - s^2 & -2sc \\ -2sc & s^2 - c^2 \end{pmatrix}$$

$$= \frac{\Delta m^2}{4p}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \equiv \vec{H}\cdot\vec{\sigma}, \quad \vec{H} = \left(\frac{\Delta m^2}{4p}\sin 2\theta,\ 0,\ \frac{-\Delta m^2}{4p}\cos 2\theta\right)$$

$$\Rightarrow U = e^{-iHt} = e^{-i\vec{H}\cdot\vec{\sigma}t}, \quad \text{use } e^{i(\vec{a}\cdot\vec{\sigma})} = \mathbb{I}\cos a + i(\hat{n}\cdot\vec{\sigma})\sin a$$

$$\vec{a} = a\hat{n} = \frac{\Delta m^2}{4p}t(-\sin 2\theta,\ 0,\ \cos 2\theta)$$

$$= \mathbb{I}\cdot\cos\left(+\frac{\Delta m^2}{4p}t\right) + i\sin\left(\frac{\Delta m^2}{4p}t\right)\cdot\left[-\sin 2\theta\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \cos 2\theta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

$$\Rightarrow P_{ee} = |a_e|^2 = |U_{11}|^2 = \left|\cos\frac{\Delta m^2}{4p}t + i\sin\frac{\Delta m^2}{4p}t\cdot\cos 2\theta\right|^2$$

$$= \cos^2\frac{\Delta m^2}{4p}t + \sin^2\frac{\Delta m^2}{4p}t\cdot\cos^2 2\theta$$

$$= 1 - \sin^2\frac{\Delta m^2}{4p}t\ \sin^2 2\theta$$

$$P_{ex} = 1 - P_{ee} = \sin^2\frac{\Delta m^2}{4p}t\ \sin^2 2\theta$$

$$\frac{\Delta m^2 t}{4E} \simeq 1.267 \times \left(\frac{\Delta m^2}{eV^2}\right)\left(\frac{L}{km}\right)\left(\frac{GeV}{E}\right)$$

Matter (electron) effects.

coherent forward scattering of neutrinos on matters

for potential egy V.

$$\because E = \sqrt{p^2 + m^2} + V \sim p + \frac{m^2}{2p} + V \quad , \text{with } e^-, p, n, \ldots$$

$$V = \underset{\substack{\swarrow \\ \nu_e \text{ only}}}{V_{\nu_e, CC}} + \underbrace{V_{\nu e, NC} + V_{\nu p} + V_{\nu n}}_{\text{neutral current } \propto \mathbb{I}} \quad , \text{no effect on oscillation.}$$

$$H^{CC}_{\substack{\text{eff.} \\ \nu_e e}} = \frac{G_F}{\sqrt{2}} [\bar{\nu}_{e,L} \gamma_\mu e_L][\bar{e}_L \gamma^\mu \nu_{e,L}]$$

$$\rightarrow V_{\nu ee} = \langle H \rangle = \sqrt{2} G_F n_e.$$ assuming n_e isotropically @ rest

Geometrical Representation – polarization vector or neutrino flavor isospin.

2-flavor :

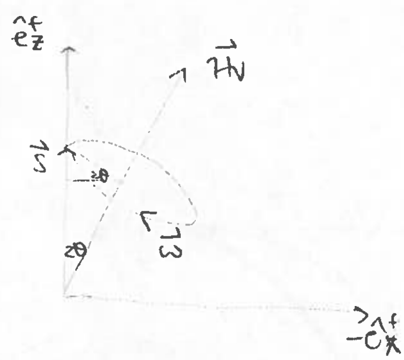
$$H_\nu \equiv \underbrace{-\frac{\vec\sigma}{2}\cdot(\omega \vec H_\nu)}, \quad \omega = \frac{\Delta m^2}{2E}, \quad \vec H_\nu = (-\sin 2\theta,\ 0,\ \cos 2\theta)$$

~ Hamiltonian of a spin in an external B field.

$$\vec S_\nu \equiv \langle \nu | \frac{\vec\sigma}{2} | \nu \rangle = \left(\mathrm{Re}(a_e^* a_x),\ \mathrm{Im}(a_e^* a_x),\ |a_e|^2 - \frac{1}{2} \right)$$

or $\dfrac{|a_e|^2 - |a_x|^2}{2}$

$$\rightarrow \quad \frac{d}{dt}\vec S = \omega \vec S \times \vec H_\nu$$



$$S_z = \frac{1}{2}\cos^2 2\theta + \frac{1}{2}\sin^2 2\theta \cdot \cos \omega t$$

$$= \frac{1}{2} - \sin^2 2\theta_V \sin^2 \frac{\omega t}{2}$$

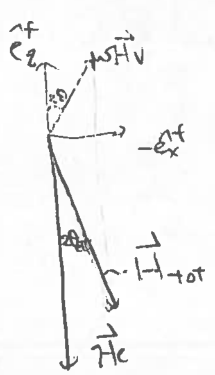
MSW resonance

$$\vec H_e = (0,\ 0,\ -\sqrt{2}G_F n_e),$$

① $\sqrt{2}G_F n_e \gg \omega$: $2\theta_{eff} \sim \dfrac{\omega \sin 2\theta}{\sqrt{2}G_F n_e} \ll 1$ → instantaneous mass state ~ flavor state.

② $\sqrt{2}G_F n_e = \omega \cos 2\theta$ → resonance, $\rho_{e,res} \sim n_e \cdot m_p \sim 260 \times \left(\frac{\mathrm{MeV}}{E}\right) \mathrm{g/cm^3}$

$\rho_{\odot,RO} \sim 100\ \mathrm{g/cc}$



③ $\sqrt{2}G_F n_e \ll \omega$, vacuum limit

adiabatic transform : $\vec H(t=0)_{\rho\to\infty} \propto -\hat e_z^f \xrightarrow{\theta \ll 1} \vec H(t\to\infty)_{\rho\to 0} \propto \hat e_z^f$

adiabaticity : $\left| \dfrac{d}{dt}\left(\dfrac{\vec H_{tot}}{|\vec H_{tot}|} \right) \right|_{res} \ll |\vec H_{tot}|_{res}$